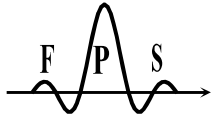


Summary

- *Frequency-domain representation of discrete signals and systems*
 - *Response of an LTI system to a complex exponential*
 - *Fourier representation of a discrete-time sequence*
- *A Review of the discrete-time Fourier Transform (DTFT)*
 - *Symmetry properties of the Fourier Transform*
 - *Theorems regarding the Fourier Transform*
 - *Table of Fourier pairs*
- *The DTFT of the auto-correlation and of the cross-correlation*
 - *the DTFT of the auto-correlation*
 - *the DTFT of the cross-correlation*
 - *examples*



Frequency-domain representation of discrete signals & systems

- **Question:** what is the output of an LTI system when the input is a complex exponential ?

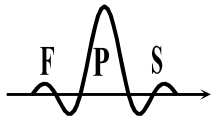
$$x[n] = e^{j\omega n}, \quad -\infty < n < +\infty$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[n]h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = \sum_{k=-\infty}^{+\infty} h[k]e^{j\omega(n-k)} = \sum_{k=-\infty}^{+\infty} h[k]e^{-j\omega k}e^{j\omega n} = H(e^{j\omega})e^{j\omega n}$$

- **Answer:** it's the complex exponential possibly modified in magnitude and phase according to the frequency response of the LTI system.
- **Note:** this result reveals that $e^{j\omega n}$ is an eigen function of the LTI system and that $H(e^{j\omega})$ is the eigen value of the system at the angular frequency ω radians.
- **Definition of the frequency response of an LTI system**
(obtained by computing the Fourier transform of its impulse response)

$$H(e^{j\omega}) \triangleq \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n} = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})}$$

- $|H(e^{j\omega})|$ → absolute value of the frequency response of the system
- $\angle H(e^{j\omega})$ → phase of the frequency response of the system



Frequency-domain representation of discrete signals & systems

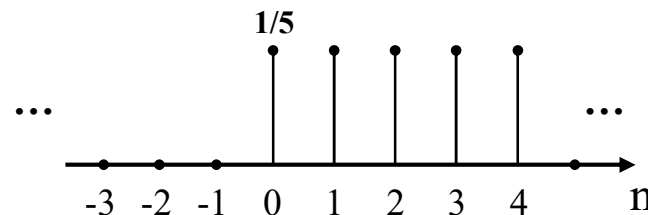
- **Example:** what is the response of an LTI system, with $h[n]$ real, to the input $x[n]=A\cos(\omega_0n+\phi)$?

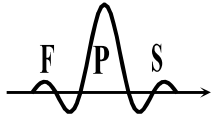
- **Answer:** $x[n]$ may be expressed in a convenient way: $x[n] = \frac{A}{2} [e^{j(\omega_0n+\phi)} + e^{-j(\omega_0n+\phi)}]$ and then:

$$y[n] = \frac{A}{2} [H(e^{j\omega_0})e^{j(\omega_0n+\phi)} + H(e^{-j\omega_0})e^{-j(\omega_0n+\phi)}] = A |H(e^{j\omega_0})| \cos[\omega_0n + \phi + \angle H(e^{j\omega_0})]$$

- Important property of $H(e^{j\omega})$
given the periodicity of the discrete complex exponential, $e^{j\omega n}$, the frequency response $H(e^{j\omega})$ is periodic with period 2π , so that in order to characterize it completely, it is sufficient to represent the magnitude and phase considering a frequency span of 2π radians, e.g., between $-\pi$ and $+\pi$ or 0 and 2π .
- Example: what is the frequency response of a moving-average filter of length 5 ?

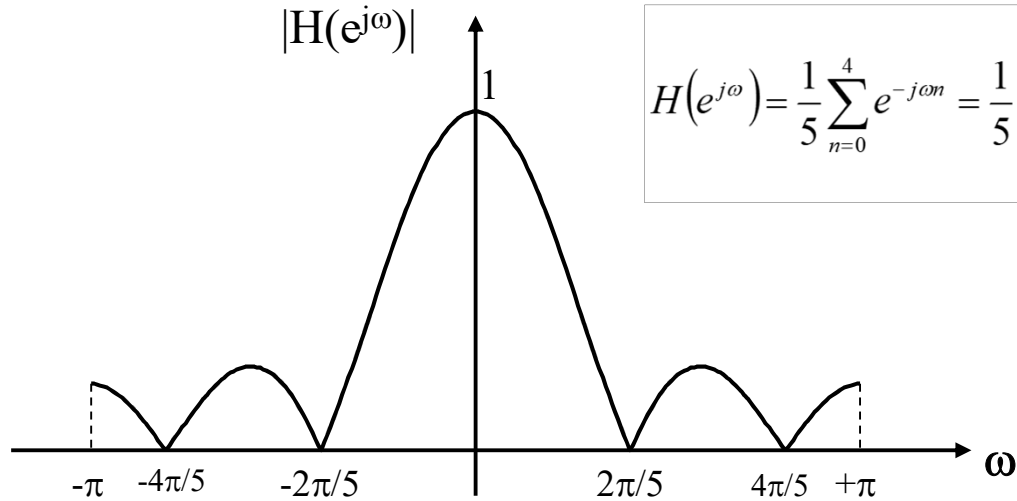
$$h[n] = \begin{cases} 1/5 & 0 \leq n \leq 4 \\ 0 & \text{outros} \end{cases}$$





Frequency-domain representation of discrete signals & systems

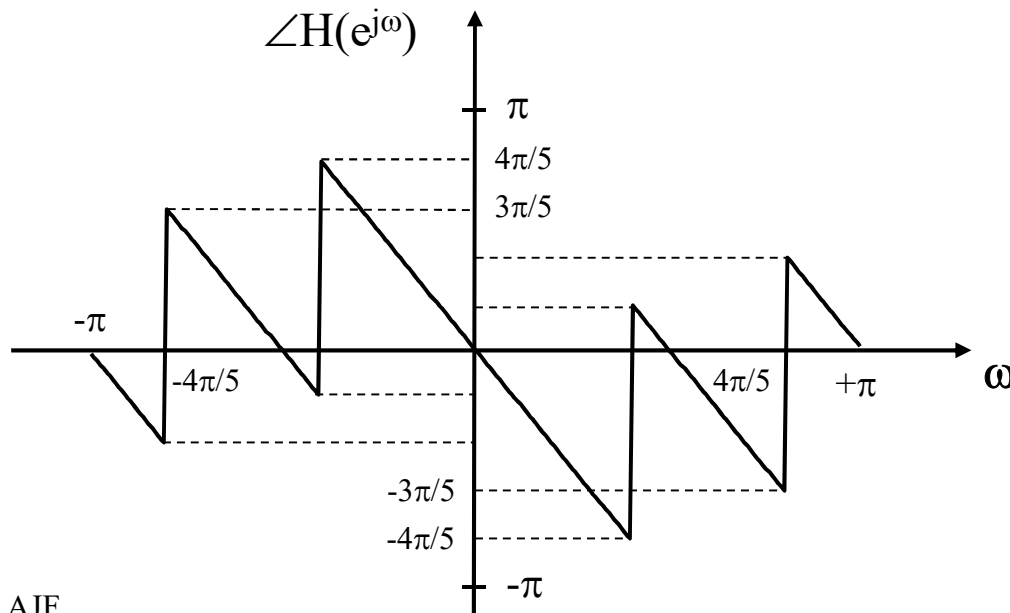
– **Answer:** using the definition of the time-discrete Fourier transform:



$$H(e^{j\omega}) = \frac{1}{5} \sum_{n=0}^4 e^{-jn\omega} = \frac{1}{5} \cdot \frac{1 - e^{-j5\omega}}{1 - e^{-j\omega}} = \frac{1}{5} e^{-j2\omega} \frac{\sin \frac{5}{2}\omega}{\sin \frac{\omega}{2}} = |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$$

NOTE 1: the magnitude function is even.

NOTE 2: the phase function is odd.

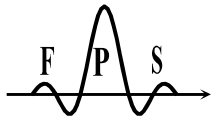


Question 1: why is that

$$\angle H(e^{j\omega}) \neq -2\omega ?$$

(note that $-1 = e^{\pm j\pi}$)

Question 2: why is that in this representation of $\angle H(e^{j\omega})$ we say that the phase is *wrapped* ?
(what is the fundamental period in the representation of phase ?)



Fourier representation of a discrete sequence

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad \xleftrightarrow{\mathbf{F}} \quad X(e^{j\omega}) = |X(e^{j\omega})| e^{j\angle X(e^{j\omega})} = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

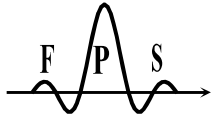
- the Fourier transform of a discrete-time signal $x[n]$ is periodic with period 2π and exists if $x[n]$ is absolutely summable
- the inverse Fourier transform allows to synthesize $x[n]$ using a period of its representation in the frequency domain

– **Example:**

$$x[n] = a^n u[n] \quad \xleftrightarrow{\mathbf{F}} \quad X(e^{j\omega}) = \sum_{n=0}^{+\infty} a^n e^{-j\omega n} = \sum_{n=0}^{+\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

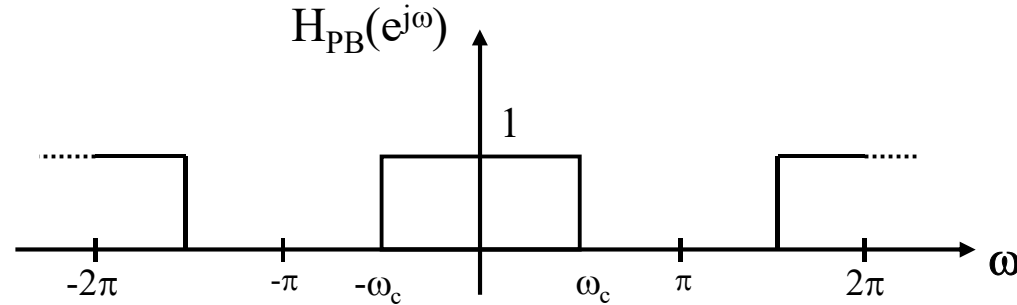
↑

if $|ae^{-j\omega}| < 1 \quad \therefore \quad |a| < 1$



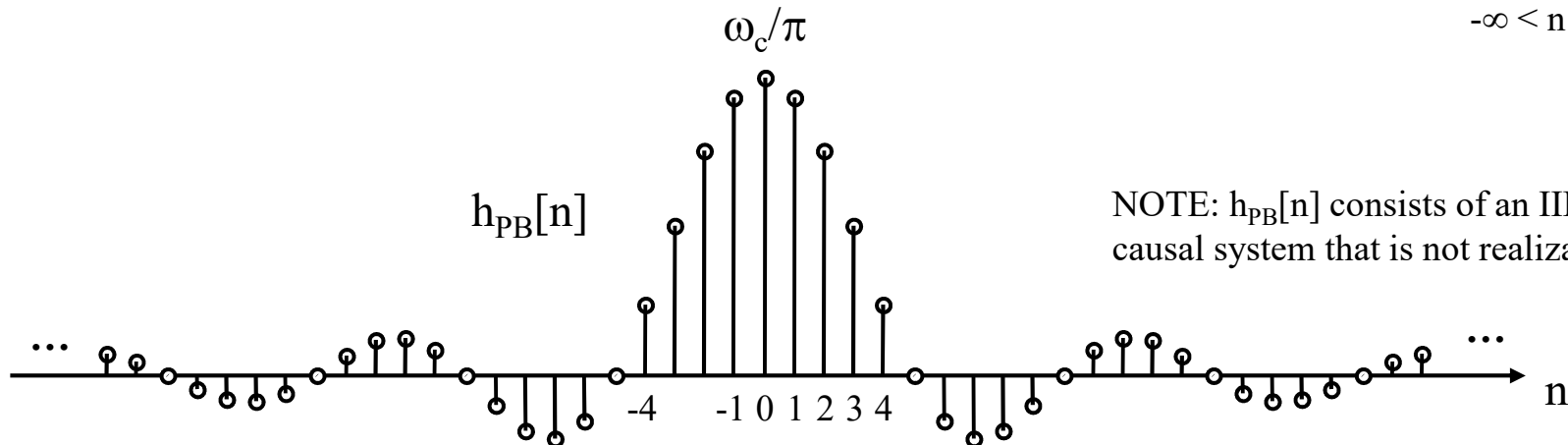
Fourier representation of a discrete sequence

- **Example:** what is the impulse response of an ideal low-pass filter ?



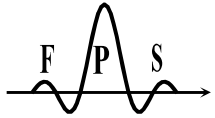
$$H_{PB}(e^{j\omega}) \xleftrightarrow{F} h_{PB}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{PB}(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{\sin n\omega_c}{n\pi}$$

$$-\infty < n < +\infty$$



NOTE: $h_{PB}[n]$ consists of an IIR non-causal system that is not realizable !

NOTE+: the response $h_{PB}[n]$ is not absolutely summable, but its square is summable, which highlights the fact that a filter resulting from $h_{PB}[n]$ by limiting its length, is the best approximation, in the mean-square sense, to $H_{PB}(e^{j\omega})$ (*i.e.* to the ideal filter).

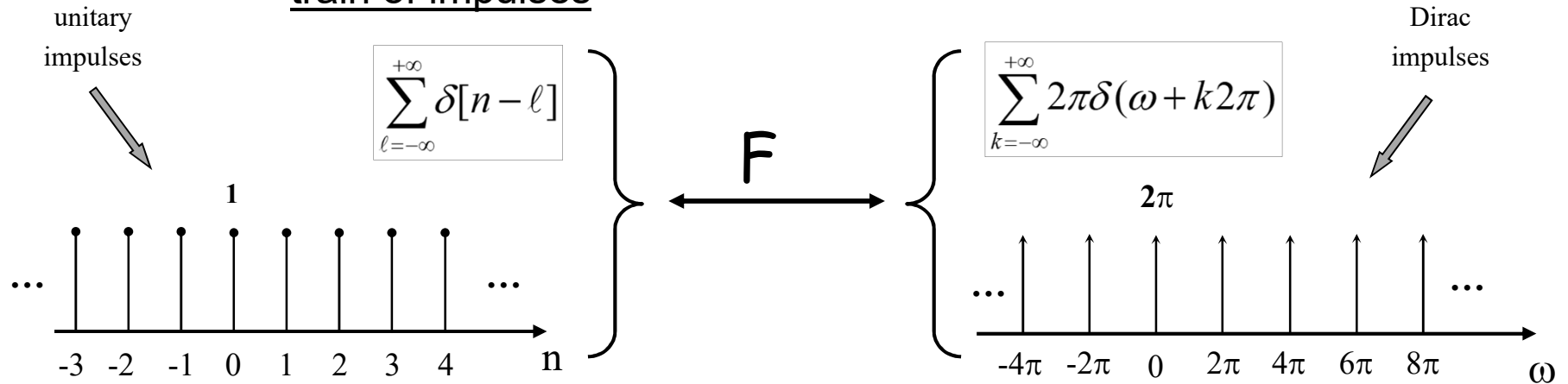


Fourier representation of a discrete sequence

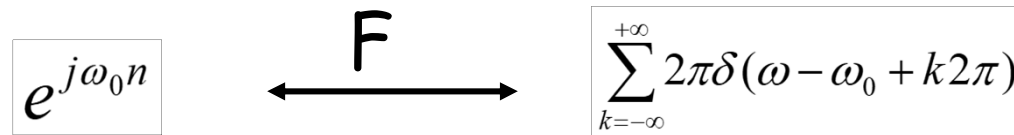
– special cases

these are special cases because they are neither absolutely summable nor square-summable, they arise from the theory of generalized functions but they are very important in the analysis of signals and discrete-time systems:

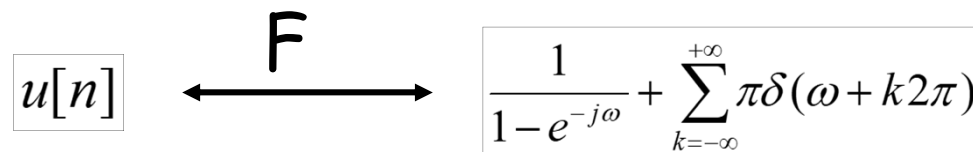
- train of impulses

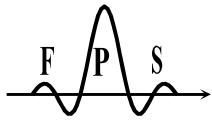


- unitary complex exponential



- unitary step





Symmetry properties of the time-discrete Fourier transform

- given $x[n]$, we may express $x[n]=x_e[n]+x_o[n]$ where:

$$x_e[n] = \frac{1}{2}(x[n] + x^*[-n]) = x_e^*[-n]$$

- $x_e[n]$ is the conjugate symmetric sequence of $x[n]$; in case $x[n]$ is real, $x_e[n]$ is also known as the *even* component of $x[n]$ since $x_e[n]=x_e[-n]$

$$x_o[n] = \frac{1}{2}(x[n] - x^*[-n]) = -x_o^*[-n]$$

- $x_o[n]$ is the conjugate anti-symmetric sequence of $x[n]$; in case $x[n]$ is real, $x_o[n]$ is also known as the *odd* component of $x[n]$ since $x_o[n]= -x_o[-n]$

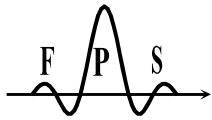
- similarly, $X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$

$$X_e(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + X^*(e^{-j\omega})] = X_e^*(e^{-j\omega})$$

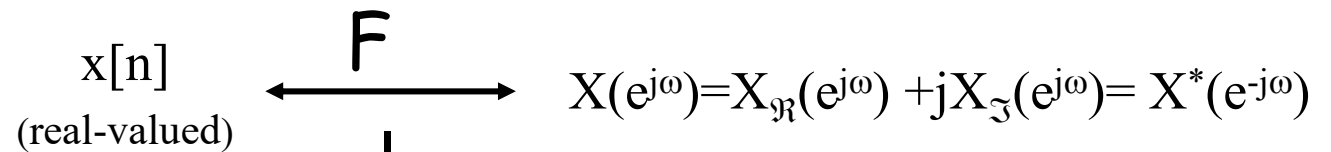
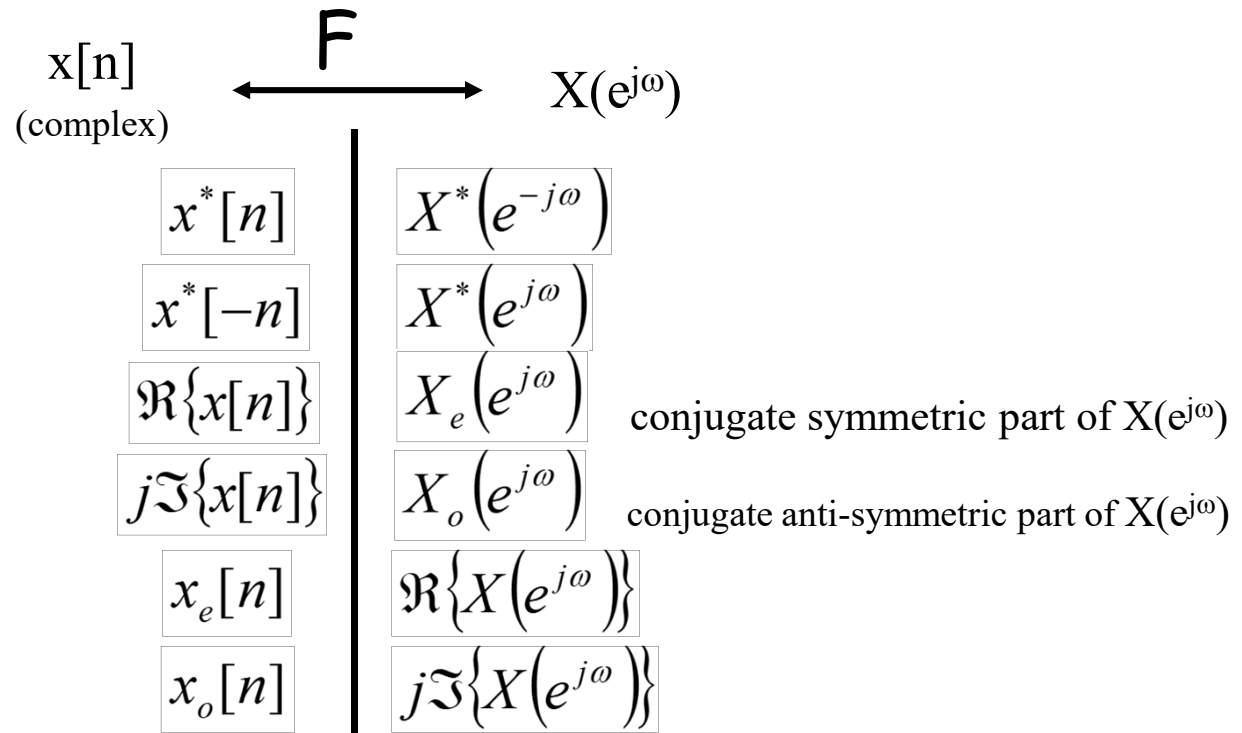
- $X_e(e^{j\omega})$ is the conjugate symmetric function of $X(e^{j\omega})$, $X_e(e^{j\omega})$ is also said the *even* component of $X(e^{j\omega})$ when $X(e^{j\omega})$ is real-valued

$$X_o(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) - X^*(e^{-j\omega})] = -X_o^*(e^{-j\omega})$$

- $X_o(e^{j\omega})$ is the conjugate anti-symmetric function of $X(e^{j\omega})$, $X_o(e^{j\omega})$ is also said the *odd* component of $X(e^{j\omega})$ when $X(e^{j\omega})$ is real-valued

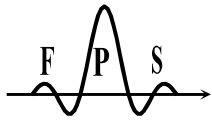


Main symmetry properties of the time-discrete Fourier transform



i.e. the transform is conjugate symmetric :

$x_e[n]$		$X_{\Re}(e^{j\omega})$	$X_{\Re}(e^{j\omega}) = X_{\Re}(e^{-j\omega})$
$x_o[n]$		$jX_{\Im}(e^{j\omega})$	$X_{\Im}(e^{j\omega}) = -X_{\Im}(e^{-j\omega})$
			$ X(e^{j\omega}) = X(e^{-j\omega}) $
			$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$



Review of the main Fourier transform theorems

(relate operations involving discrete sequences and the corresponding operations in the Fourier domain)

$$x[n], y[n] \xleftrightarrow{F} X(e^{j\omega}), Y(e^{j\omega})$$

linearity

$$ax[n] + by[n]$$

$$aX(e^{j\omega}) + bY(e^{j\omega})$$

shift in n

$$x[n - n_d]$$

$$e^{-j\omega n_d} X(e^{j\omega})$$

n_d inteiro

shift in ω

$$e^{j\omega_0 n} x[n]$$

$$X[e^{j(\omega - \omega_0)}]$$

'time' reversal

$$x[-n]$$

$$X(e^{-j\omega})$$

differentiation in ω

$$nx[n]$$

$$j \frac{dX(e^{j\omega})}{d\omega}$$

why is there no
"differentiation" in n ?

convolution

$$x[n] * y[n]$$

$$X(e^{j\omega}) \cdot Y(e^{j\omega})$$

product

$$x[n] \cdot y[n]$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j(\omega - \theta)}) d\theta$$

(periodic convolution)

Parseval theorem

$$\sum_{n=-\infty}^{+\infty} x[n] \cdot y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) Y^*(e^{j\omega}) d\omega$$

Parseval theorem
(particular case)

$$E = \sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

energy

energy spectral density

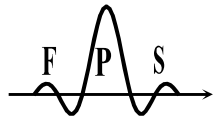


Tabela de pares de Fourier

example:

$$a^n u[n], \quad |a| < 1$$

$$x[n] \xrightarrow{\text{F}} X(e^{j\omega})$$

$$\frac{1}{1 - ae^{-j\omega}}$$

$$\delta[n]$$

$$1$$

$$\delta[n - n_0]$$

$$e^{-j\omega n_0}$$

$$\sum_{\ell=-\infty}^{+\infty} \delta[n - \ell]$$

$$\sum_{k=-\infty}^{+\infty} 2\pi \delta(\omega + k2\pi)$$

$$e^{j\omega_0 n}$$

$$\sum_{k=-\infty}^{+\infty} 2\pi \delta(\omega - \omega_0 + k2\pi)$$

$$u[n]$$

$$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega + k2\pi)$$

$$(n+1)a^n u[n], \quad |a| < 1$$

$$1/|1 - ae^{-j\omega}|^2$$

$$\begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{outros} \end{cases}$$

$$\frac{\sin(M+1)\frac{\omega}{2}}{\sin\frac{\omega}{2}} \cdot e^{-j\omega\frac{M}{2}}$$

$$\cos(\omega_0 n + \phi)$$

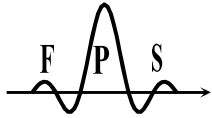
$$\pi \sum_{k=-\infty}^{+\infty} [e^{j\phi} \delta(\omega - \omega_0 + k2\pi) + e^{-j\phi} \delta(\omega + \omega_0 + k2\pi)]$$

$$\frac{\sin n\omega_c}{n\pi}$$

$$\begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

$$r^n \frac{\sin \omega_p (n+1)}{\sin \omega_p} u[n], \quad |r| < 1$$

$$1/(1 - 2r \cos \omega_p e^{-j\omega} + r^2 e^{-j2\omega})$$



Question: what is a practical way to find the inverse Fourier transform ?

• **Example:** $X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})}$, causal \xleftrightarrow{F} $x[n]=?$

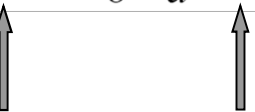
if $M < N$ and poles are first-order, then:

$$X(e^{j\omega}) = \frac{\prod_{\ell=1}^M (1 - c_{\ell} e^{-j\omega})}{\prod_{k=1}^N (1 - d_k e^{-j\omega})} = \sum_{k=1}^N \frac{A_k}{1 - d_k e^{-j\omega}}$$

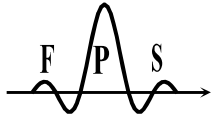
with : $A_k = (1 - d_k e^{-j\omega}) X(e^{j\omega}) \Big|_{e^{j\omega}=d_k}$

and thus: $\frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})} = \frac{a/(a-b)}{1-ae^{-j\omega}} + \frac{b/(b-a)}{1-be^{-j\omega}}$

which leads to: $x(n) = \frac{a}{a-b} a^n u[n] + \frac{b}{b-a} b^n u[n]$



Not to forget !



The DTFT of the auto-correlation and of the cross-correlation

- the DTFT of the auto-correlation

the auto-correlation is defined as (in this discussion, we admit energy signals)

$$r_x[\ell] = x[\ell] * x^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k] x^*[k - \ell]$$

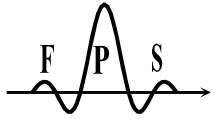
considering the DTFT properties

$$\begin{aligned} x[\ell] &\xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \\ x^*[\ell] &\xleftrightarrow{\mathcal{F}} X^*(e^{-j\omega}) \\ x[-\ell] &\xleftrightarrow{\mathcal{F}} X(e^{-j\omega}) \\ x^*[-\ell] &\xleftrightarrow{\mathcal{F}} X^*(e^{j\omega}) \end{aligned}$$

then

$$r_x[\ell] = x[\ell] * x^*[-\ell] \xleftrightarrow{\mathcal{F}} R_x(e^{j\omega}) = X(e^{j\omega}) \cdot X^*(e^{j\omega}) = |X(e^{j\omega})|^2$$

Where $R_x(e^{j\omega}) = |X(e^{j\omega})|^2$ is called the spectral density of energy



The DTFT of the auto-correlation and of the cross-correlation

- the DTFT of the auto-correlation (cont.)
 - the Wiener-Khinchine Theorem: the auto-correlation and the spectral density of energy form a Fourier pair

$$r_x[\ell] \xleftrightarrow{\mathcal{F}} R_x(e^{j\omega}) = |X(e^{j\omega})|^2$$

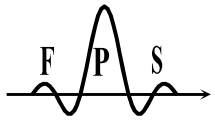
thus,

$$r_x[\ell] = \frac{1}{2\pi} \int_{-\pi}^{\pi} R(e^{j\omega}) e^{j\omega\ell} d\omega$$

and, in particular, the energy of the signal can be found using

$$E = r_x[0] = \sum_{k=-\infty}^{+\infty} |x[k]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} R(e^{j\omega}) d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

which reflects the Parseval Theorem



The DTFT of the auto-correlation and of the cross-correlation

- the DTFT of the cross-correlation
the cross-correlation is defined as (we admit energy signals)

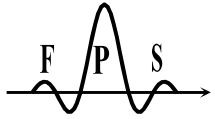
$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k] y^*[k - \ell]$$

considering the DTFT properties

$$\begin{aligned} x[\ell] &\xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \\ y[\ell] &\xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) \\ y^*[\ell] &\xleftrightarrow{\mathcal{F}} Y^*(e^{-j\omega}) \\ y[-\ell] &\xleftrightarrow{\mathcal{F}} Y(e^{-j\omega}) \\ y^*[-\ell] &\xleftrightarrow{\mathcal{F}} Y^*(e^{j\omega}) \end{aligned}$$

then

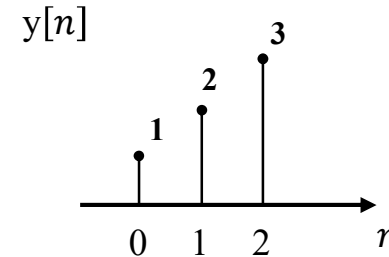
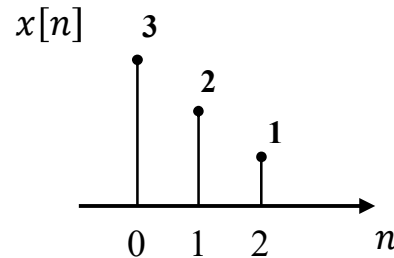
$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] \xleftrightarrow{\mathcal{F}} R_{xy}(e^{j\omega}) = X(e^{j\omega}) \cdot Y^*(e^{j\omega})$$



The DTFT of the auto-correlation and of the cross-correlation

- examples

let us admit two discrete-time signals, $x[n]$ and $y[n]$



it can be easily concluded that

$$x[\ell] = 3\delta[\ell] + 2\delta[\ell - 1] + \delta[\ell - 2] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) = 3 + 2e^{-j\omega} + e^{-j2\omega}$$

$$y[\ell] = \delta[\ell] + 2\delta[\ell - 1] + 3\delta[\ell - 2] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = 1 + 2e^{-j\omega} + 3e^{-j2\omega}$$

$$R_x(e^{j\omega}) = 3e^{j2\omega} + 8e^{j\omega} + 14 + 8e^{-j\omega} + 3e^{-j2\omega} = R_y(e^{j\omega}), \text{ (why ?)}$$

$$R_{xy}(e^{j\omega}) = 9e^{j2\omega} + 12e^{j\omega} + 10 + 4e^{-j\omega} + e^{-j2\omega}$$