

FORMULAE SHEET

| | |
|---------------------------|--|
| Laplace Transform | $X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st} dt$ $x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st} ds$ $e^{-\alpha t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s + \alpha}, \quad \Re\{s\} > -\alpha$ |
| Discrete-time convolution | $x[n] * h[n] = \sum_{\ell=-\infty}^{+\infty} x[\ell]h[n - \ell] = \sum_{\ell=-\infty}^{+\infty} h[\ell]x[n - \ell] = h[n] * x[n]$ |
| Fourier Transform | $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ $x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n} d\omega$ $x[-n] \xleftrightarrow{\mathcal{F}} X(e^{-j\omega})$ $x^*[n] \xleftrightarrow{\mathcal{F}} X^*(e^{-j\omega})$ $x_e[n] = \frac{x[n] + x^*[-n]}{2} = x_e^*[-n] \xleftrightarrow{\mathcal{F}} \Re\{X(e^{j\omega})\}$ $x_o[n] = \frac{x[n] - x^*[-n]}{2} = -x_o^*[-n] \xleftrightarrow{\mathcal{F}} j\Im\{X(e^{j\omega})\}$ $\Re\{x[n]\} \xleftrightarrow{\mathcal{F}} X_e(e^{j\omega}) = X_e^*(e^{-j\omega})$ $j\Im\{x[n]\} \xleftrightarrow{\mathcal{F}} X_o(e^{j\omega}) = -X_o^*(e^{-j\omega})$ $x[n - n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} X(e^{j\omega})$ $e^{jn\omega_0} x[n] \xleftrightarrow{\mathcal{F}} X(e^{j(\omega - \omega_0)})$ $nx[n] \xleftrightarrow{\mathcal{F}} j \frac{dX(e^{j\omega})}{d\omega}$ $x[n] * y[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})Y(e^{j\omega})$ |
| (periodic convolution) | $x[n]y[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$ $e^{jn\omega_0} \xleftrightarrow{\mathcal{F}} 2\pi \sum_{k=-\infty}^{+\infty} \delta(\omega - \omega_0 + k2\pi)$ $\sum_{\ell=-\infty}^{+\infty} \delta[n - \ell] \xleftrightarrow{\mathcal{F}} 2\pi \sum_{k=-\infty}^{+\infty} \delta(\omega + k2\pi)$ $\delta[n] \xleftrightarrow{\mathcal{F}} 1$ $\delta[n - n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0}$ $a^n u[n], \quad a < 1 \xleftrightarrow{\mathcal{F}} \frac{1}{1 - ae^{-j\omega}}$ $(n + 1)a^n u[n], \quad a < 1 \xleftrightarrow{\mathcal{F}} \frac{1}{ 1 - ae^{-j\omega} ^2}$ |
| Low-pass filter | $\frac{\sin(n\omega_c)}{n\pi} \xleftrightarrow{\mathcal{F}} \begin{cases} 1, & \omega \leq \omega_c \\ 0, & \omega_c < \omega \leq \pi \end{cases}$ |
| Parseval Theorem | $\sum_{n=-\infty}^{+\infty} x[n]y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$ |
| Cross-correlation | $r_{xy}[\ell] = x[\ell] * y^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]y^*[k - \ell]$ $r_{xy}[\ell] \xleftrightarrow{\mathcal{F}} R_{xy}(e^{j\omega}) = X(e^{j\omega})Y^*(e^{j\omega})$ $r_{xy}[\ell] = r_{yx}^*[-\ell]$ $ r_{xy}[\ell] \leq \sqrt{r_x[0]r_y[0]}$ |

Auto-correlation

$$r_x[\ell] = x[\ell] * x^*[-\ell] \xleftrightarrow{\mathcal{F}} R_x(e^{j\omega}) = |X(e^{j\omega})|^2$$

$$r_x[\ell] = r_x^*[-\ell] \quad |r_x[\ell]| \leq r_x[0]$$

Ideal sampling

$$x_c(t) \xleftrightarrow{\mathcal{F}} X_c(\Omega) \quad x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$x[n] = x_c(nT) \rightarrow X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c\left(\Omega - k\frac{2\pi}{T}\right), \quad \omega = \Omega T$$

Ideal reconstruction

$$y_c(t) = \sum_{n=-\infty}^{+\infty} y[n] \operatorname{sinc}\left(\frac{\pi}{T}(t - nT)\right), \quad \operatorname{sinc}(\theta) = \frac{\sin(\theta)}{\theta}$$

Z-Transform

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}, \quad \operatorname{RoC} \equiv C_x \quad x[n] = \frac{1}{2\pi j} \oint X(z)z^{n-1} dz$$

$$x[n - n_0] \xleftrightarrow{z} z^{-n_0} X(z), \quad \operatorname{RoC} \equiv C_x$$

$$z_0^n x[n] \xleftrightarrow{z} X(z/z_0), \quad \operatorname{RoC} \equiv |z_0|C_x$$

$$x^*[n] \xleftrightarrow{z} X^*(z^*), \quad \operatorname{RoC} \equiv C_x$$

$$x[-n] \xleftrightarrow{z} X(1/z), \quad \operatorname{RoC} \equiv 1/C_x$$

$$nx[n] \xleftrightarrow{z} -z \frac{dX(z)}{dz}, \quad \operatorname{RoC} \equiv C_x$$

$$x[n] * y[n] \xleftrightarrow{z} X(z)Y(z), \quad \operatorname{RoC} \equiv C_x \cap C_y$$

(complex convolution)

$$x[n]y^*[n] \xleftrightarrow{z} \frac{1}{2\pi j} \oint X(v)Y^*(z^*/v^*)v^{-1} dv, \quad \operatorname{RoC} \equiv C_x C_y$$

$$a^n u[n] \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$-a^n u[-n - 1] \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$

$$na^n u[n] \xleftrightarrow{z} \frac{az^{-1}}{(1 - az^{-1})^2}, \quad |z| > |a|$$

$$a^n \sin(n\omega_0) u[n] \xleftrightarrow{z} \frac{a \sin(\omega_0) z^{-1}}{1 - 2a \cos(\omega_0) z^{-1} + a^2 z^{-2}}, \quad |z| > |a|$$

$$a^n \cos(n\omega_0) u[n] \xleftrightarrow{z} \frac{1 - a \cos(\omega_0) z^{-1}}{1 - 2a \cos(\omega_0) z^{-1} + a^2 z^{-2}}, \quad |z| > |a|$$

Parseval Theorem

$$\sum_{n=-\infty}^{+\infty} x[n]y^*[n] = \frac{1}{2\pi j} \oint X(z)Y^*(1/z^*)z^{-1} dz, \quad \operatorname{RoC} \equiv C_x C_y$$

Correlation in Z-domain

$$r_x[\ell] = x[\ell] * x^*[-\ell] \xleftrightarrow{z} R_x(z) = X(z)X^*\left(\frac{1}{z^*}\right), \quad \operatorname{RoC} \equiv C_x \cap 1/C_x$$

$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] \xleftrightarrow{z} R_{xy}(z) = X(z)Y^*\left(\frac{1}{z^*}\right), \quad \operatorname{RoC} \equiv C_x \cap 1/C_y$$

Residues (1st-order poles)

$$X(z) = \sum_{k=1}^N \frac{A_k}{1 - d_k z^{-1}} \quad A_k = (1 - d_k z^{-1})X(z)|_{z=d_k}$$

Frequency Response and Group Delay

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})} \quad \tau(\omega) = -\frac{d}{d\omega}\{\angle H(e^{j\omega})\}$$

All-pass (gain normalized)

$$H(z) = \frac{z^{-1} - \alpha^*}{1 - \alpha z^{-1}} \quad H(z) = \frac{1}{|\alpha|} \frac{z - \frac{1}{\alpha^*}}{z - \alpha}$$

Linear-phase system

$$H(e^{j\omega}) = A(\omega)e^{j(\beta - \alpha\omega)}; \quad A(\omega) \text{ is real-valued; } \beta, \alpha \text{ are constants}$$

IIR filter (causal)

$$y[n] = \sum_{k=0}^{M-1} b_k x[n-k] + \sum_{\ell=1}^{N-1} a_\ell y[n-\ell]$$

Butterworth filter

$$H_b(s)H_b(-s) = \frac{1}{1 + \left(\frac{s}{j\Omega_c}\right)^{2N}} \quad |H_b(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$s_k = \Omega_c e^{j\frac{\pi}{2N}(2k+N+1)}, \quad k = 0, 1, \dots, 2N-1$$

Impulse Invariance

$$h[n] = T h_c(nT) \quad H(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} H_c\left(\frac{\omega - k2\pi}{T}\right)$$

$$H_C(s) = \sum_{k=0}^{N-1} \frac{A_k}{s - s_k} \quad H(z) = T \sum_{k=0}^{N-1} \frac{A_k}{1 - e^{s_k T} z^{-1}}$$

Bilinear

Transformation

$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \quad z = \frac{2 + sT}{2 - sT}$$

$$\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right) \quad \omega = 2 \tan^{-1}\left(\frac{\Omega T}{2}\right)$$

Rectangular window

$$w_R[n] = u[n] - u[N] = \begin{cases} 1, & n = 0, 1, \dots, N-1 \\ 0, & \text{other } n \end{cases}$$

Generalized Hamming window

$$w_H[n] = \left(\alpha - (1 - \alpha) \cos\left(\frac{2\pi n}{N-1}\right)\right) w_R[n]$$

$$\text{Hamming: } \alpha = 0.54$$

$$\text{Hanning: } \alpha = 0.50$$

Window method (FIR)

$$H_d(e^{j\omega}) \xrightarrow{F} h_d[n] \quad h[n] = h_d[n]w[n]$$

$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} H_d(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta$$

Parks-McClellan
(Linear-phase FIR)

$$H(e^{j\omega}) = A(\omega)e^{-j(\alpha\omega-\beta)}; \quad A(\omega) \text{ is real-valued; } \beta, \alpha \text{ are constants}$$

$$A(\omega) = Q(\omega) \sum_{n=0}^r a[n](\cos \omega)^n$$

$$Q(\omega) = 1 : (\text{FIR type 1}) \quad Q(\omega) = \cos\left(\frac{\omega}{2}\right) : (\text{FIR type 2})$$

$$Q(\omega) = \sin(\omega) : (\text{FIR type 3}) \quad Q(\omega) = \sin\left(\frac{\omega}{2}\right) : (\text{FIR type 4})$$

Sampling the Fourier
Transform

$$Y[k] = X(z)|_{z=e^{jk\frac{2\pi}{N}}} = X(e^{j\omega})|_{\omega=k\frac{2\pi}{N}}, \quad k = 0, 1, \dots, N-1$$

$$x[n] \xrightarrow{Z} X(z) \quad y[n] = \sum_{\ell=-\infty}^{+\infty} x[n + \ell N]$$

Transfer function
reconstruction

$$H(z) = \frac{1 - z^{-1}}{N} \sum_{k=0}^{N-1} \frac{H[k]}{1 - W_N^{-k} z^{-1}}, \quad W = e^{-j2\pi}$$

DFT ($W_\beta^\alpha = e^{-j2\pi\frac{\alpha}{\beta}}$)

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

$$x[(-n)_N] \xrightarrow{DFT} X[(-k)_N] \quad x^*[n] \xrightarrow{DFT} X^*[(-k)_N]$$

$$x_{ep}[n] = \frac{x[n] + x^*[(-n)_N]}{2} = x_{ep}^*[-n] \xrightarrow{DFT} \Re\{X[k]\}$$

$$x_{op}[n] = \frac{x[n] - x^*[(-n)_N]}{2} = -x_{op}^*[-n] \xrightarrow{DFT} j\Im\{X[k]\}$$

$$\Re\{x[n]\} \xrightarrow{DFT} X_{ep}(e^{j\omega})$$

$$\begin{aligned}
 j\mathfrak{S}\{x[n]\} &\xleftrightarrow{DFT} X_{op}(e^{j\omega}) \\
 x[(n - n_0)_N] &\xleftrightarrow{DFT} e^{-j\frac{2\pi}{N}kn_0}X[k] \\
 e^{j\frac{2\pi}{N}nk_0}x[n] &\xleftrightarrow{DFT} X[(k - k_0)_N] \\
 x[n] \otimes h[n] &\xleftrightarrow{DFT} X[k]H[k] \quad x[n]h[n] \xleftrightarrow{DFT} \frac{1}{N}X[k] \otimes H[k] \\
 x[n] \otimes h[n] &= \sum_{\ell=0}^{N-1} x[\ell]h[(n - \ell)_N]
 \end{aligned}$$

Decimation $d[n] = x[nM]$ $D(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(W_M^k z^{\frac{1}{M}}\right)$

Interpolation $c[n] = \begin{cases} x\left[\frac{n}{L}\right], & n \text{ multiple of } L \\ 0, & \text{other } n \end{cases}$ $C(z) = X(z^L)$

FFT DIT $X[k] = G[k] + W_N^k H[k]$ $X[k + N/2] = G[k] - W_N^k H[k]$
 FFT DIF $g[n] = x[n] + x[n + N/2]$ $h[n] = (x[n] - x[n + N/2])W_N^n$

Arithmetic/Geometric progression $\sum_{k=1}^M k = \frac{M(M+1)}{2}$ $\sum_{k=0}^M \alpha^k = \frac{1 - \alpha^{M+1}}{1 - \alpha}$

Sum of squares $\sum_{k=1}^M k^2 = \frac{M(M+1)(2M+1)}{6}$

Trigonometric identities

$$\begin{aligned}
 \cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\
 \sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha) \\
 \sin(\alpha) + \sin(\beta) &= 2 \sin\left(\frac{1}{2}(\alpha + \beta)\right) \cos\left(\frac{1}{2}(\alpha - \beta)\right) \\
 \cos(\alpha) + \cos(\beta) &= 2 \cos\left(\frac{1}{2}(\alpha + \beta)\right) \cos\left(\frac{1}{2}(\alpha - \beta)\right)
 \end{aligned}$$

| θ | 0 | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ |
|----------------|---|--------------|--------------|--------------|----------|
| $\sin(\theta)$ | 0 | 1/2 | $\sqrt{2}/2$ | $\sqrt{3}/2$ | 1 |
| $\cos(\theta)$ | 1 | $\sqrt{3}/2$ | $\sqrt{2}/2$ | 1/2 | 0 |
| $\tan(\theta)$ | 0 | $\sqrt{3}/3$ | 1 | $\sqrt{3}$ | ∞ |