# Summary

- *Sampling and reconstruction of continuous signals*
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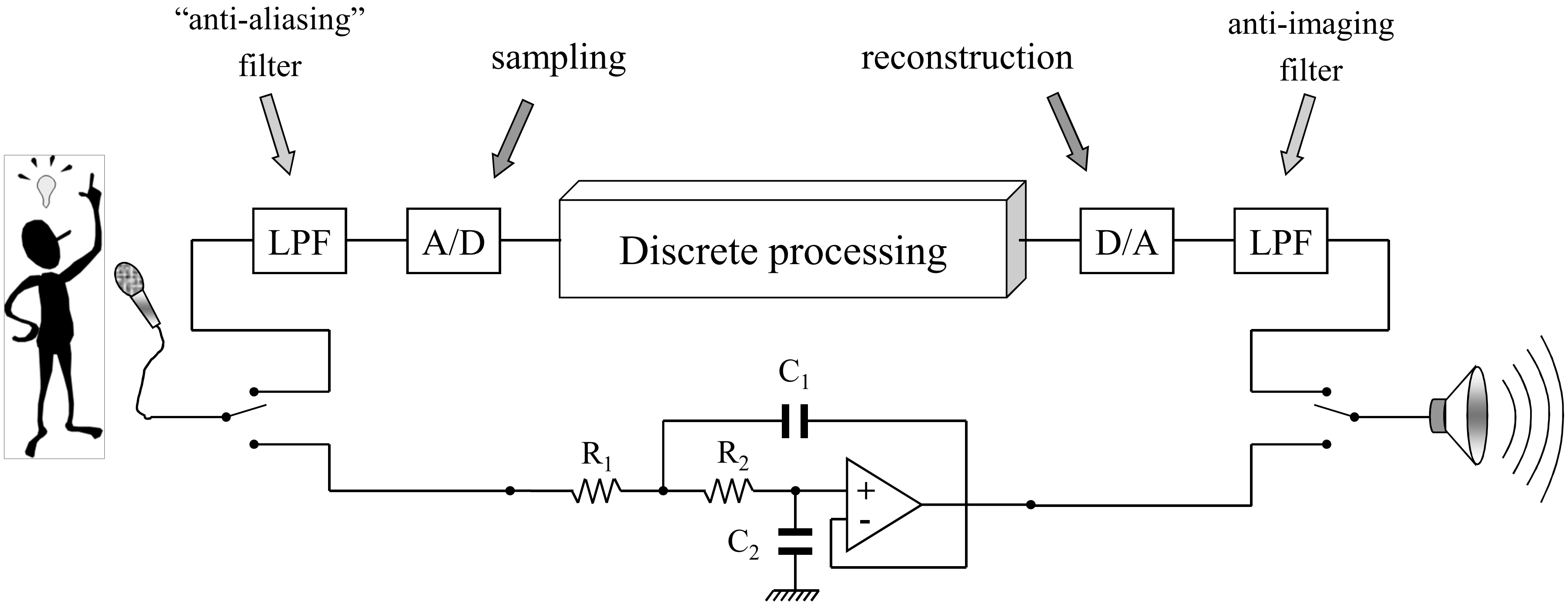
- Introduction
  - most discrete-time signals result from sampling (*i.e.* discretization in time) of continuous-time signals
  - under certain conditions, a discrete-time signal may be an exact representation (*i.e.* there is no loss of information) of a continuous-time signal
  - any form of processing of a continuous-time signal may be realized in the discrete domain, which requires the sampling of the continuous-time signal before processing, and the reconstruction of the continuous-time signal from samples after the processing stage

# Sampling of continuous-time signals

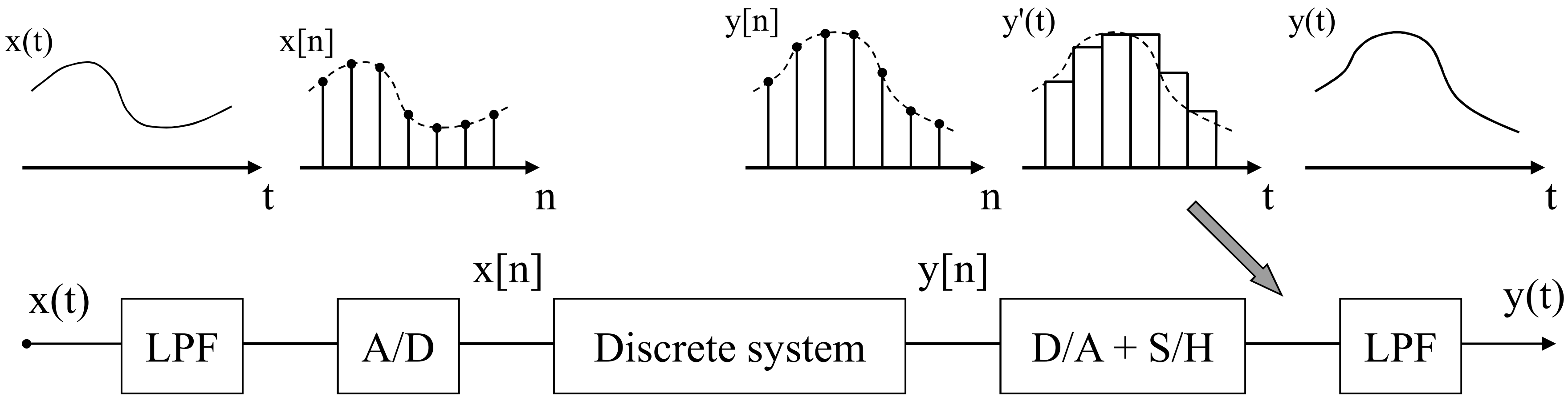
- Introduction (cont.)
  - is discrete-time processing preferable to analog processing ?




  - e.g. is there any non-trivial analog filter with exact linear phase ? (but easy to realize using a discrete-time system…)

# Sampling of continuous-time signals

- Context
  - minimal structure for the discrete-time processing of analog signals:




  - in the following we admit that the sampling rate is constant and that the A/D and D/A converters have infinite resolution (i.e., no quantization errors)
  - QUESTION: in the absence of discrete-time processing, i.e., if y[n]=x[n], and admitting ideal A/D and D/A converters, under which conditions is it possible to sample and reconstruct an analog signal without loss of information, i.e., such that y(t)=x(t) ?

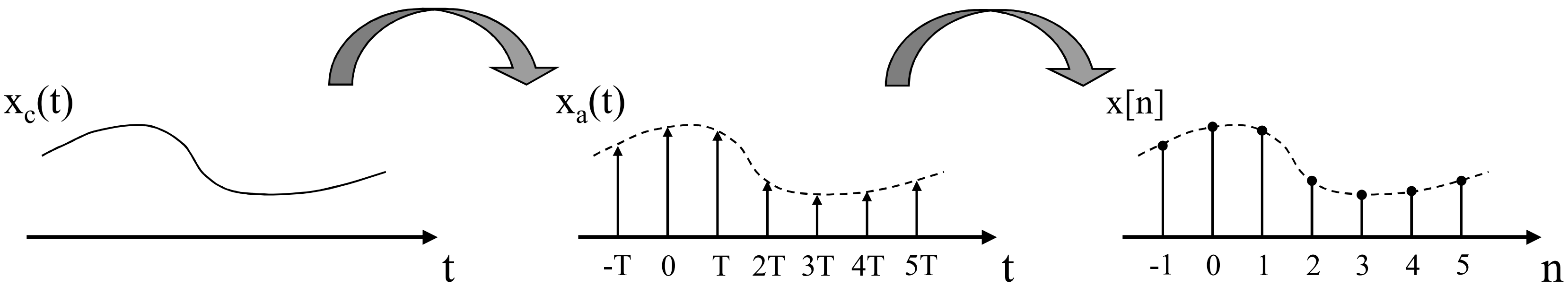
# Sampling of continuous-time signals

– in order to answer the previous question, we analyze two fundamental steps in the represented block diagram : the time discretization of the continuous-time signal by means of a periodic sampling (continuous-time signal → discrete-time signal conversion) and the time reconstruction of the continuous-time signal from samples (discrete-time signal → continuous-time signal conversion)

- periodic sampling

$$x_a(t) = x_c(t) \times s(t)$$

$$x[n] = x_a(nT)$$




T: sampling period (sec.)

$$x_a(t) = x_c(t) \times s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t - nT)$$

1/T: sampling frequency (Hertz)

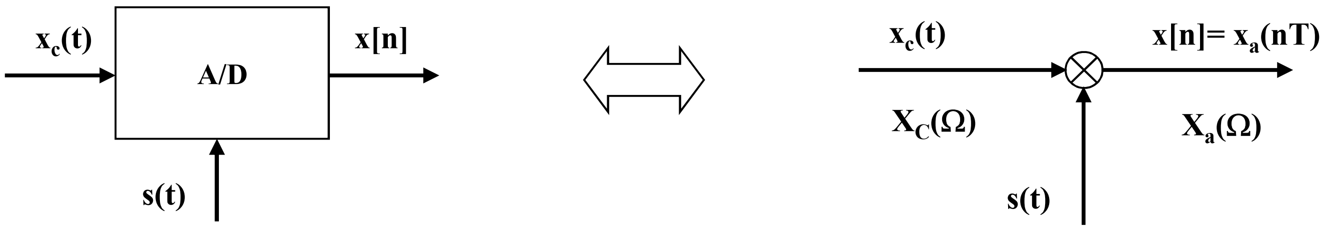
$\Omega_s = 2\pi/T$: angular sampling frequency (radians/seg.)

$$x[n] = x_a(nT) \quad , \quad -\infty < n < +\infty$$

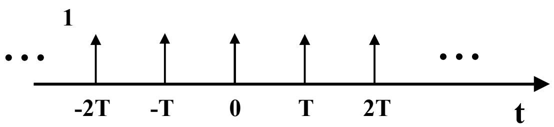
– NOTE: this operation is only invertible (*i.e.,* the ambiguity is avoided of two different signals giving rise to the same discrete signal) if $x_c(t)$ is constrained.

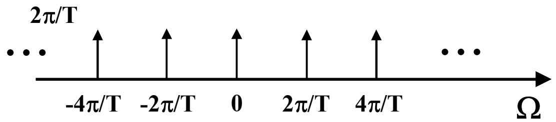
# Frequency domain analysis of periodic sampling

– time discretization: how to relate $X(e^{j\omega})$ and $X_c(\Omega)$ ?




$$s(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT) \quad \overset{F}{\longleftrightarrow} \quad S(\Omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\Omega-k\frac{2\pi}{T}\right)$$




$$x_a(t)=x_c(t)\cdot s(t)=\sum_{n=-\infty}^{+\infty}x_c(nT)\delta(t-nT) \quad \overset{F}{\longleftrightarrow} \quad X_a(\Omega)=\frac{1}{2\pi}X_c(\Omega)*S(\Omega)=\frac{1}{T}\sum_{k=-\infty}^{+\infty}X_c\left(\Omega-k\frac{2\pi}{T}\right)$$

and also:

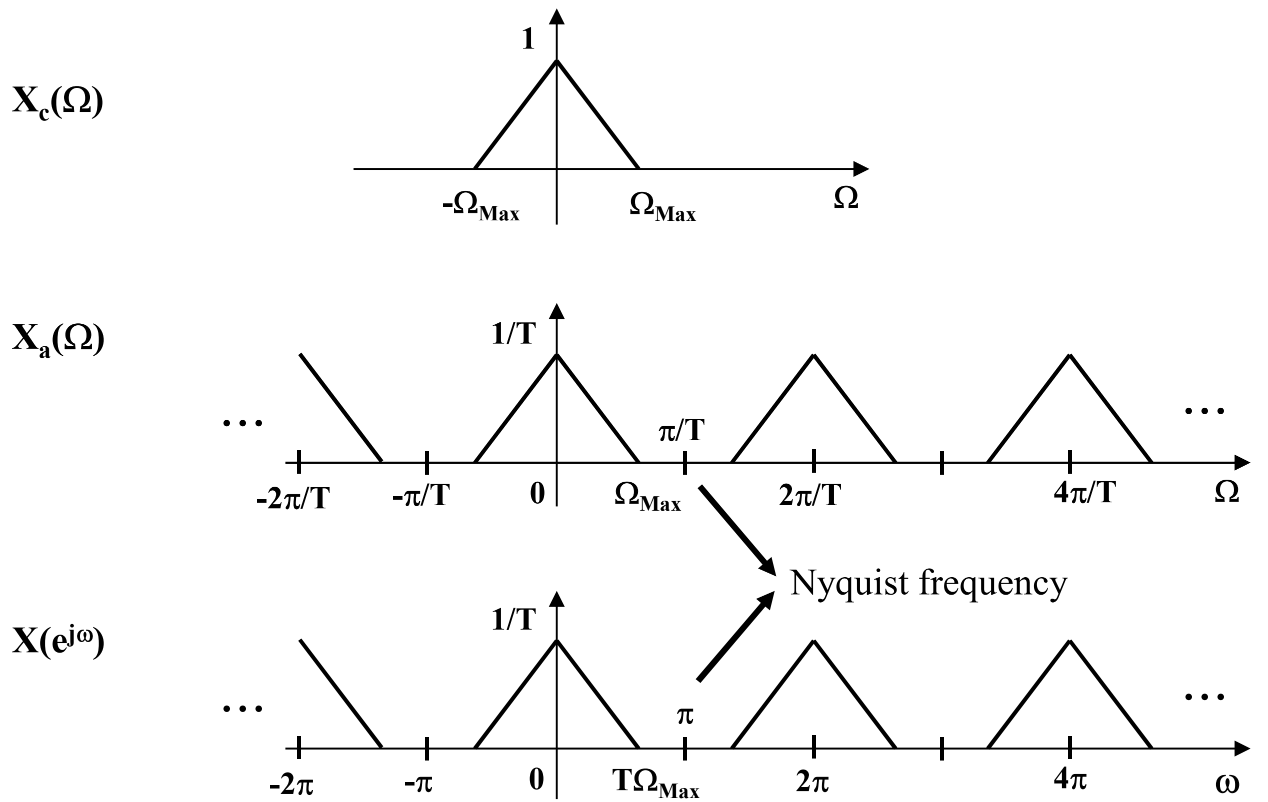
$$x_a(t)=\sum_{n=-\infty}^{+\infty}x_c(nT)\delta(t-nT) \quad \overset{F}{\longleftrightarrow} \quad X_a(\Omega)=\sum_{n=-\infty}^{+\infty}x_c(nT)e^{-jn\Omega T}\Bigg|_{\substack{x[n]=x_c(nT)\\ \omega=\Omega T}}=\sum_{n=-\infty}^{+\infty}x[n]e^{-jn\omega}=X\left(e^{j\omega}\right)$$

thus:

$$X(e^{j\omega})=X_a(\Omega)\Big|_{\Omega=\frac{\omega}{T}}=\frac{1}{T}\sum_{k=-\infty}^{+\infty}X_c\left(\frac{\omega-k2\pi}{T}\right)$$

# Frequency domain analysis of periodic sampling

→ The previous result says that except for a scale factor and a normalization (by 1/T) of the frequency axis (making that the "analog" frequency $k2\pi/T=k\Omega_s$ be projected in the "digital" frequency $k2\pi$, for any integer K) the spectra $X(e^{j\omega})$ and $X_a(\Omega)$ are similar. It also says that, as result of ideal sampling, the spectrum of the continuous-time signal appears replicated at all multiple integers of the sampling frequency.



# Frequency domain analysis of periodic sampling

The Nyquist sampling theorem

- in order to avoid spectral overlap (i.e., *aliasing*) between replicas of the base-band spectrum, it must be ensured that :

$$\Omega_{MAX} < \pi/T = \Omega_S/2 \quad \Leftrightarrow \quad 2\pi F_{MAX} < \pi F_S \quad \Leftrightarrow \quad F_S > 2F_{MAX}$$

- this means that the bandwidth of the base-band signal must be limited to less than half the sampling frequency. This condition is typically enforced by a low-pass filter just before the A/D converter, thus named "anti-aliasing" filter.

- if this condition is guaranteed, as the illustration suggests, it is possible to recover $X_c(\Omega)$ from $X(e^{j\omega})$, using an ideal low-pass continuous-time filter, with gain T and cut-off frequency $\Omega_{MAX} < \Omega_p < \Omega_S - \Omega_{MAX}$

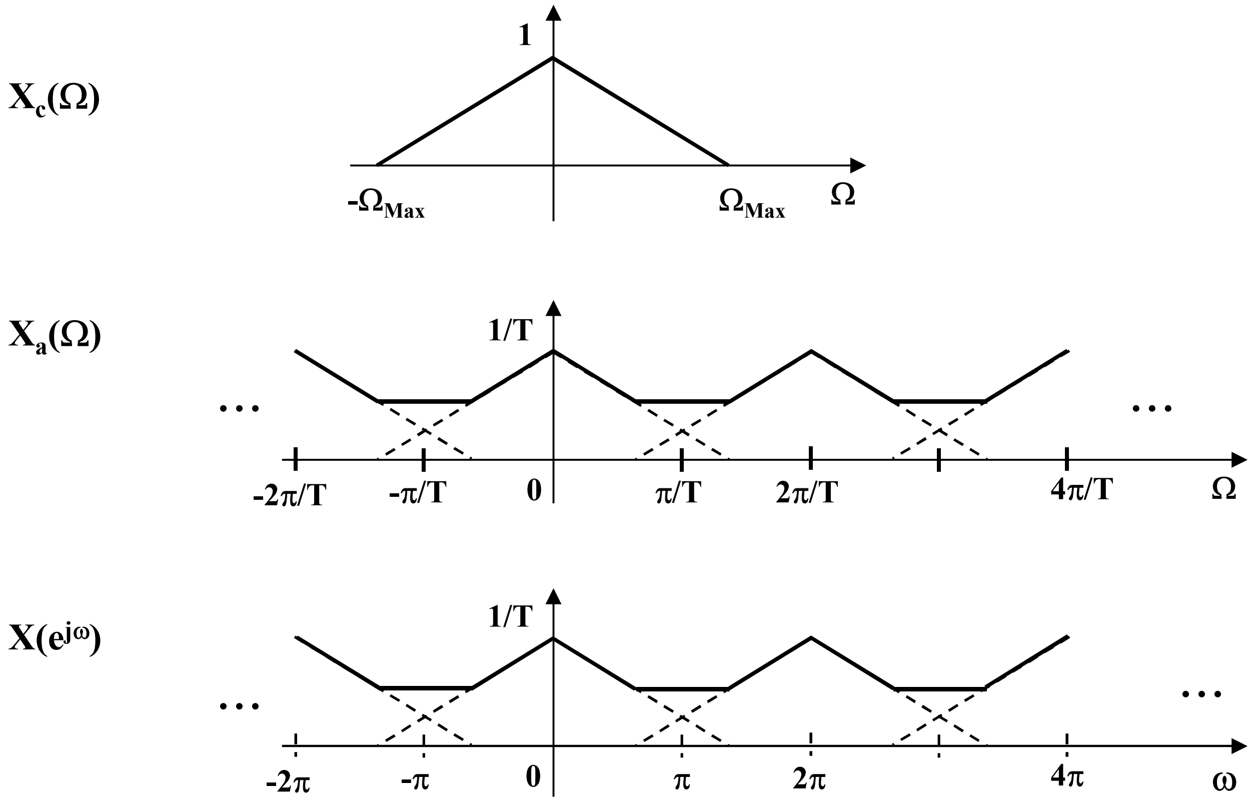
these aspects reflect the Nyquist sampling theorem:

- is $x_c(t)$ is a band-limited signal such that $X_c(\Omega) = 0$ for $|\Omega| > \Omega_{MAX}$, then $x_c(t)$ is uniquely determined (*i.e.* may be unambiguously reconstructed) from its samples $x[n]=x_c(nT)$ with $\Omega_S=2\pi/T > 2\Omega_{MAX}$

NOTE: $\Omega_S/2=\pi/T$ is commonly known as the Nyquist frequency.

# Frequency domain analysis of periodic sampling

→ what if the sampling condition is violated, i.e., if $F_S < 2F_{MAX}$ ?

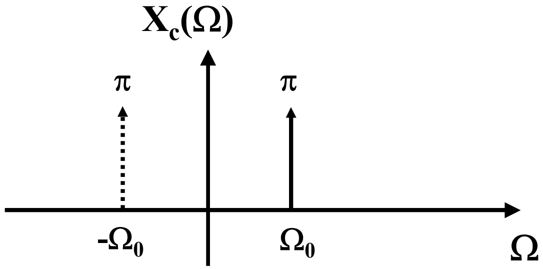


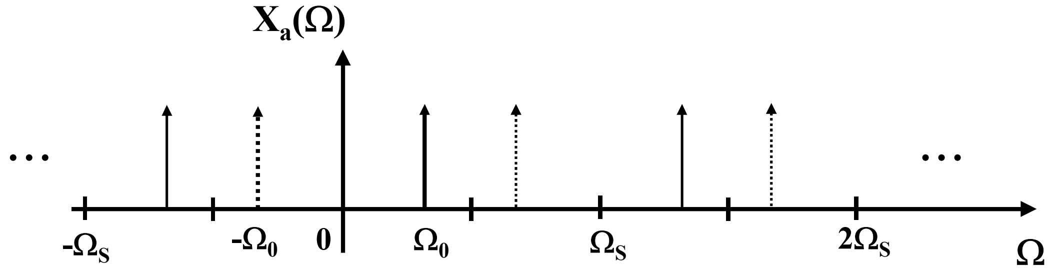

**Answer:** there is spectral overlap ( “aliasing” ) distorting the signal, and preventing the recovery of the original spectrum after low-pass filtering.

# Frequency domain analysis of periodic sampling

example: case of a continuous-time signal (co-sinusoidal function) correctly and incorrectly sampled

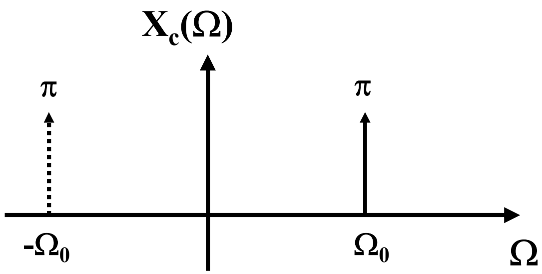
$x_c(t)=\cos(\Omega_0 t)$ , $\Omega_0 < \Omega_S/2$ $\therefore$ there is no "aliasing"

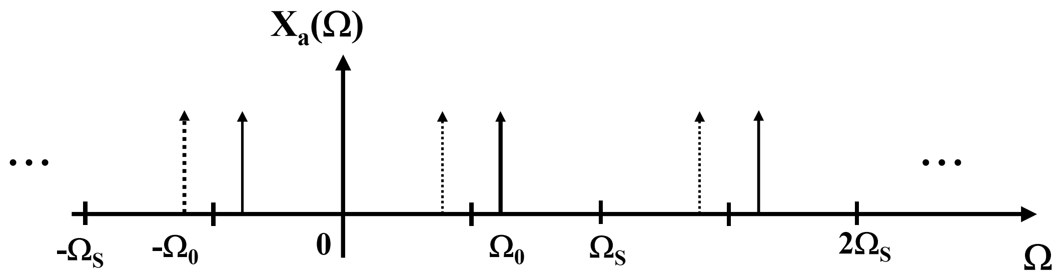



recovered signal after low-pass filtering, with cut-off at $\Omega_S/2$ : $x_c(t)=\cos(\Omega_0 t)$

$x_c(t)=\cos(\Omega_0 t)$ , $\Omega_0 > \Omega_S/2$ $\therefore$ there is "aliasing"

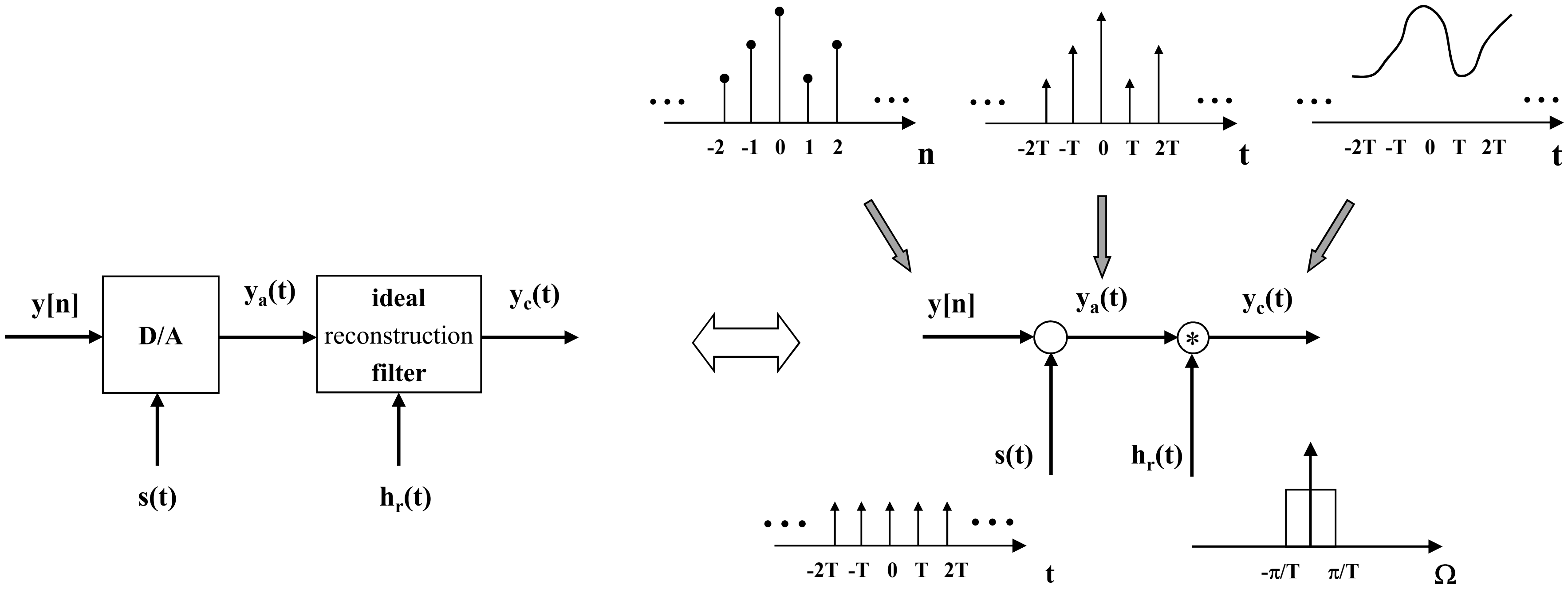



recovered signal after low-pass filtering, with cut-off at $\Omega_S/2$ : $x_c(t)=\cos[(\Omega_S-\Omega_0)t]$

# Reconstruction from samples

- Case 1: ideal reconstruction
  - as can be concluded from the spectral representation of $X_a(\Omega)$ ('slide' nº 7), if we preserve solely the base-band replica after low-pass filtering, then it is possible to recover the spectrum $X_c(\Omega)$; the same is to say: it is possible to recover $x_c(t)$. This is the principle which we will illustrate next using y[n].

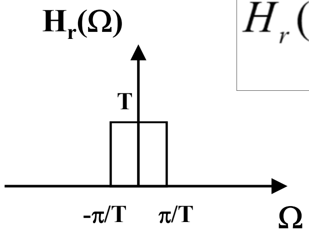


# Reconstruction from samples

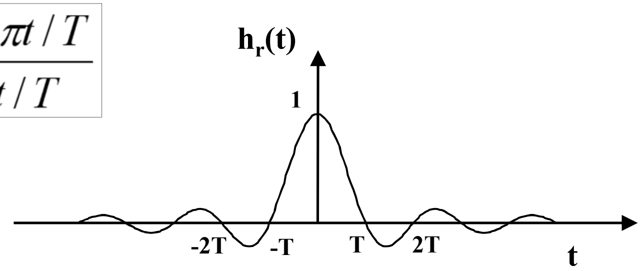
The first step going from the discrete-time domain to the continuous-time domain involves placing the pulses of the discrete sequence y[n] at instants uniformly distributed in time, thus obtaining $y_a(t)$. It should be noted that this signal has the same spectrum of $x_a(t)$ since we presume that y[n]=x[n].

$$y_a(t) = \sum_{n=-\infty}^{+\infty} y[n]\delta(t-nT) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y_a(\Omega) = \sum_{n=-\infty}^{+\infty} y_a(nT)e^{-jn\Omega T}\Big|_{\substack{y(n)=y_a(nT)\\ \omega=\Omega T}} = \sum_{n=-\infty}^{+\infty} y[n]e^{-jn\Omega T} = Y\left(e^{j\Omega T}\right)$$

By submitting the continuous-time signal $y_a(t)$ to an ideal low-pass filter having impulse response $h_r(t)$, gain T and cutting-off frequency at $\pi$/T:




$$H_r(\Omega) = \begin{cases} T, & |\Omega| < \pi/T \\ 0, & other \end{cases} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad h_r(t) = \frac{\sin \pi t/T}{\pi t/T}$$



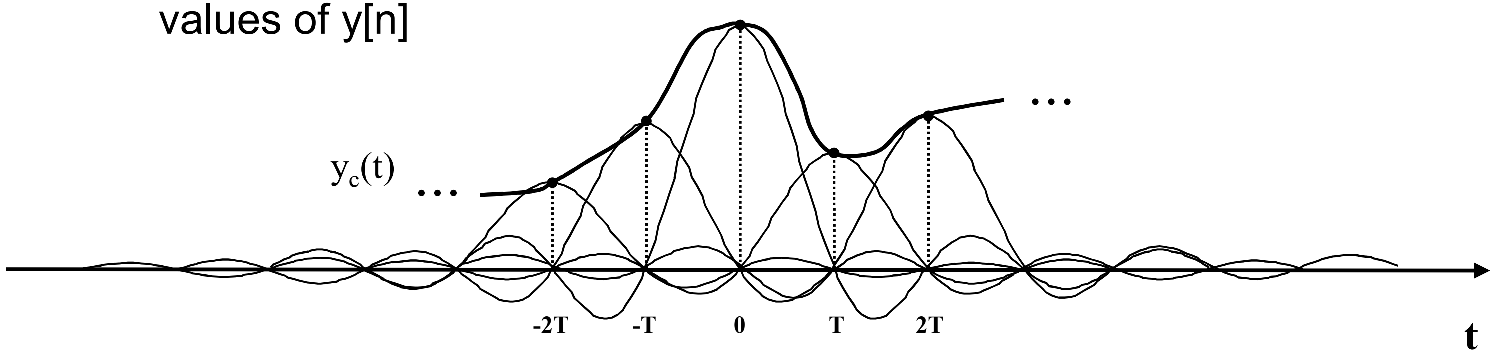

we obtain:

$$y_c(t) = y_a(t) * h_r(t) = \sum_{n=-\infty}^{\infty} y_a(nT)\frac{\sin \pi(t-nT)/T}{\pi(t-nT)/T} = \sum_{n=-\infty}^{\infty} y[n]\,\mathrm{sinc}\,\frac{\pi}{T}(t-nT)$$

# Reconstruction from samples

This result reveals that:

- at the sampling instants $y_c(nT)=y[n]=x[n]=x_c(nT)$, given that all sinc functions in the summation are zero, except one (that centered at t=nT) whose value is 'one',
- at intermediary instants, the continuous-time signal results from the sum of all sinc functions, i.e. the filter $h_r(t)$ implements an interpolation using all values of y[n]




using frequency-domain analysis, and considering y[n]=x[n] which implies: $Y_a(\Omega)=Y\left(e^{j\omega}\right)\Big|_{\omega=\Omega T}=X\left(e^{j\omega}\right)\Big|_{\omega=\Omega T}=X_a(\Omega)$

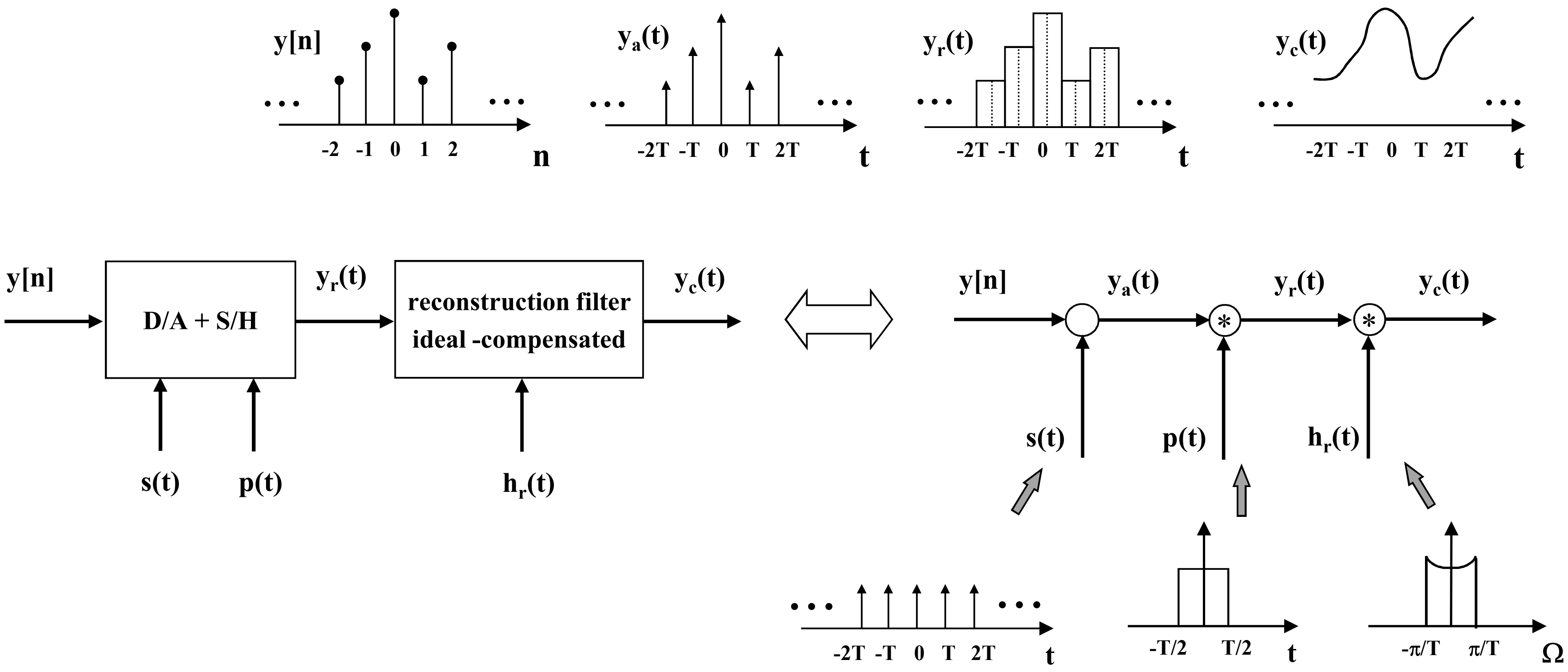
It can be concluded that the result of filtering is:

$$y_c(t)=y_a(t)*h_r(t) \quad \overset{F}{\longleftrightarrow} \quad Y_c(\Omega)=X_a(\Omega)\cdot H_r(\Omega)=\frac{1}{T}\sum_{k=-\infty}^{+\infty}X_c\left(\Omega-k\frac{2\pi}{T}\right)H_r(\Omega)=X_c(\Omega)$$

which means that, considering ideal conditions and the Nyquist criterion, it is possible to reconstruct the continuous-time signal from its samples, without loss of information. **Question**: the reconstruction filter is also known as anti-imaging filter, why ?

# Reconstruction from samples

- Case 2: zero-order real reconstruction
  - real electronic devices, in particular D/A converters, do not operate using pulses but use instead more physically tractable signals such as boxcar function approximations. Let us consider the case closest to reality where the D/A converter is associated with a “sample-and-hold” device that ‘retains’ the value of a sample during a sampling period, giving rise to a staircase-like signal:



# Reconstruction from samples

as considered before:

$$y_a(t) = \sum_{n=-\infty}^{+\infty} y[n]\delta(t-nT) \quad \xleftrightarrow{\;F\;} \quad Y_a(\Omega) = Y(e^{j\omega})\Big|_{\omega=\Omega T} = Y(e^{j\Omega T})$$

and for the boxcar function of width T:

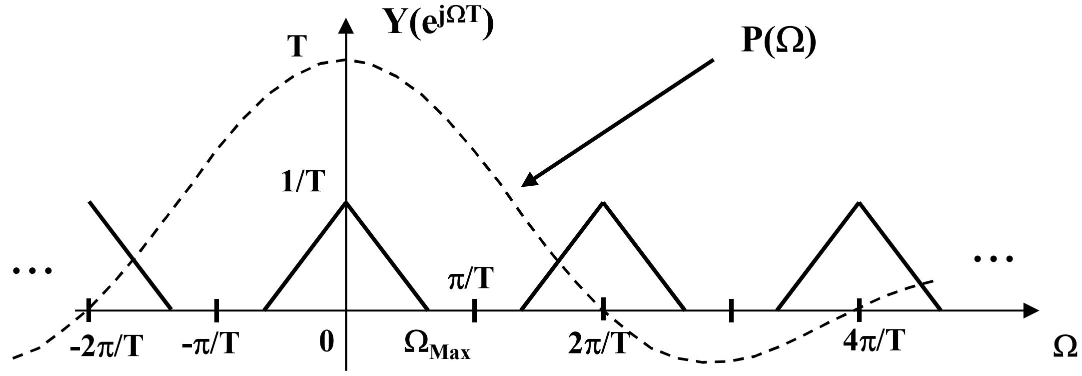
$$p(t) = \begin{cases} 1, & |t| < T/2 \\ 0, & outros \end{cases} \quad \xleftrightarrow{\;F\;} \quad P(\Omega) = T\frac{\sin \Omega T/2}{\Omega T/2}$$

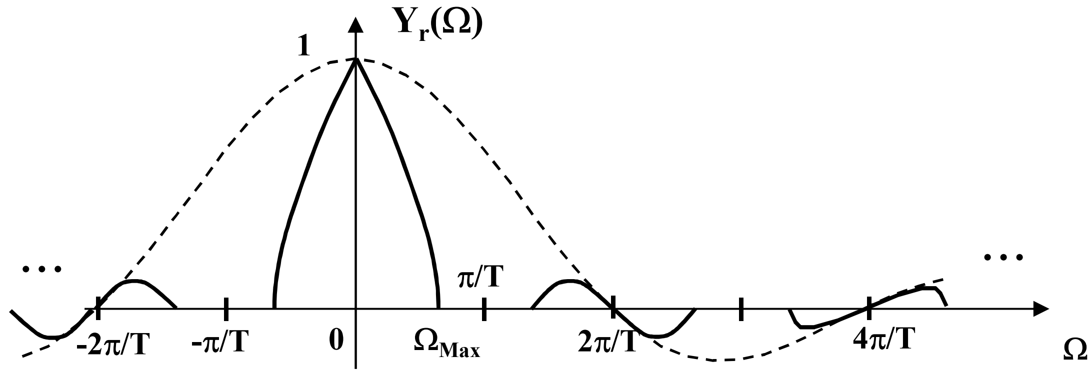
and therefore $y_r(t)$ results as:

$$y_r(t) = y_a(t) * p(t) = \sum_{n=-\infty}^{+\infty} y[n]p(t-nT) \quad \xleftrightarrow{\;F\;} \quad Y_r(\Omega) = Y(e^{j\Omega T})T\frac{\sin \Omega T/2}{\Omega T/2}$$

# Reconstruction from samples

whose spectral representation is:




from where it can be concluded that the zero-order reconstruction distorts the $Y(e^{j\Omega T})$ spectrum in a way that can be compensated for, if we consider the base-band replica which is the one we want to recover; in addition, all other replicas which we want to eliminate, are strongly attenuated which alleviates the filtering effort of $h_r(t)$.

# Reconstruction from samples

The filter $h_r(t)$ must then not only reject the undesirable spectral images, but also compensate the magnitude distortion affecting the base-band replica :

$$y_c(t) = y_r(t) * h_r(t) \quad \overset{F}{\longleftrightarrow} \quad Y_c(\Omega) = Y_r(\Omega)\cdot H_r(\Omega) = Y(e^{j\Omega T})T\cdot\frac{\sin \Omega T/2}{\Omega T/2}\cdot H_r(\Omega)$$

presuming also that y[n]=x[n], then: $Y\left(e^{j\omega}\right)\Big|_{\omega=\Omega T} = X\left(e^{j\omega}\right)\Big|_{\omega=\Omega T} = X_a(\Omega)$

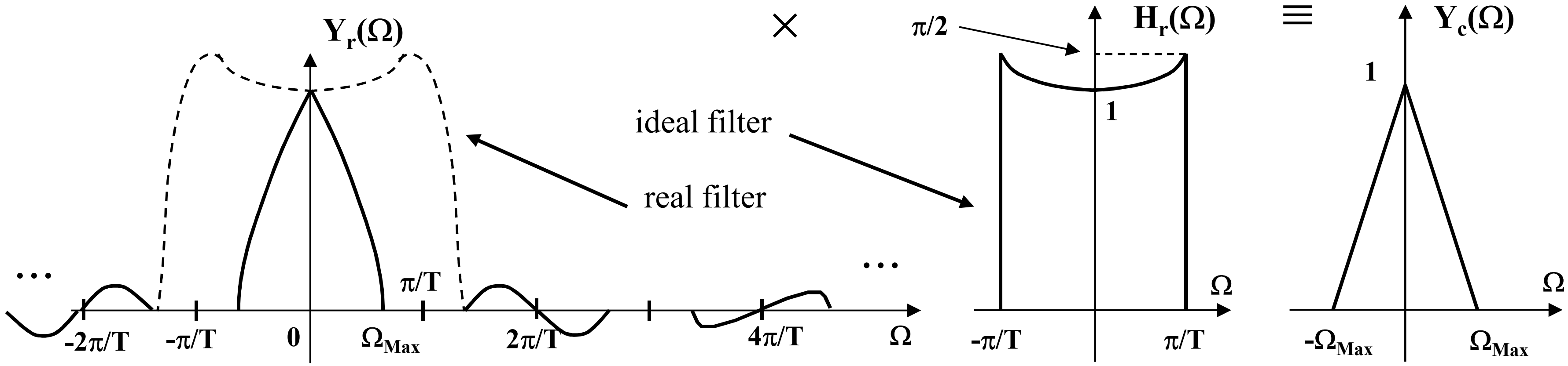
$$Y_c(\Omega) = X_a(\Omega)\cdot T\frac{\sin \Omega T/2}{\Omega T/2}\cdot H_r(\Omega) = \sum_{k=-\infty}^{+\infty} X_c\left(\Omega - k\frac{2\pi}{T}\right)\cdot\frac{\sin \Omega T/2}{\Omega T/2}\cdot H_r(\Omega) = X_c(\Omega)$$

subject to the condition that filter $H_r(\Omega)$ is low-pass, with cut-off frequency at $\pi$/T, but is also compensated such as to reverse the sin(x)/x distortion, i.e. :

$$H_r(\Omega) = \begin{cases} \dfrac{\Omega T/2}{\sin \Omega T/2}, & |\Omega| < \pi/T \\ 0, & other \end{cases}$$

# Reconstruction from samples

Then, it results graphically:




which means the output of $h_r(t)$ is also given by:

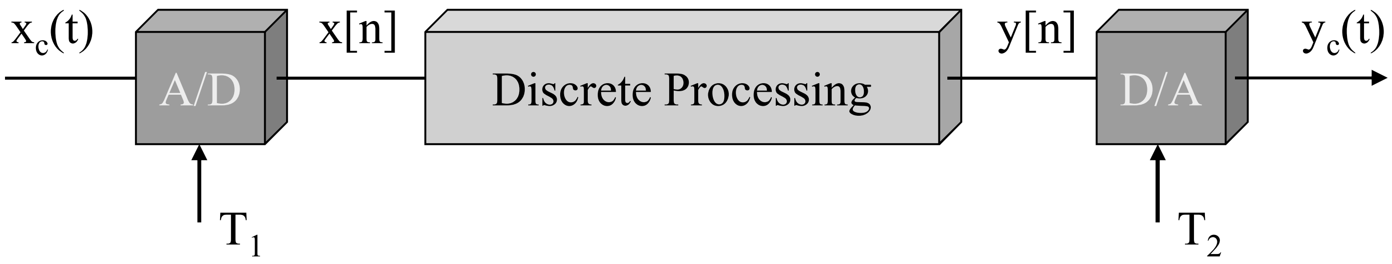
$$y_c(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\,\frac{\sin \pi (t-nT)/T}{\pi (t-nT)/T}$$

as we have already concluded before.

NOTE 1: the compensation sin(x)/x may be inserted at any stage of the processing, including (and perhaps preferably ! ) at the discrete processing stage, with all the known advantages.

NOTE 2: in addition to the 'zero-order' reconstruction, there are other possibilities (*e.g.* the 'one-order' reconstruction) !

– In our previous analysis we have admitted y[n]=x[n], i.e., absence of discrete-time processing so as to show the possibility of sampling and reconstruction an analog signal. It is important to assess now the impact on the analog signal of a discrete-time processing as this is the most common scenario:




– Although it is possible/desirable to design systems where the A/D sampling frequency is different from the D/A sampling frequency, (*e.g.* that is the case of oversampling that is used in CD/MP3 players), we admit in this analysis that both are equal.

– If the discrete-time system is LTI and is characterized in the frequency by $H(e^{j\omega})$, then: $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$

but since: $X\left(e^{j\omega}\right) = X_a(\Omega)\big|_{\Omega=\omega/T}$ which means: $X(e^{j\Omega T}) = X_a(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\left(\Omega - k\frac{2\pi}{T}\right)$

We have also seen that considering for example zero-order reconstruction, then: $Y_c(\Omega) = Y(e^{j\Omega T})T\frac{\sin \Omega T/2}{\Omega T/2}H_r(\Omega)$

and we obtain finally:

$$Y_c(\Omega) = H\left(e^{j\Omega T}\right)\cdot\frac{\sin \Omega T/2}{\Omega T/2}\cdot H_r(\Omega)\sum_{k=-\infty}^{+\infty} X_c\left(\Omega - k\frac{2\pi}{T}\right)$$

we may thus conclude that:

- if the 'anti-aliasing' filter at the input of the system enforces $X_c(\Omega)=0$ for $|\Omega|>\pi/T$ (or if $x_c(t)$ possesses this property already), then there is no overlap of spectral images in the summation
- if the reconstruction filter eliminates spectral images for $|\Omega|>\pi/T$ and ensures sin(x)/x compensation, then the previous expression simplifies to:

$$Y_c(\Omega) = H\left(e^{j\Omega T}\right)X_c(\Omega) = H_{eff}(\Omega)X_c(\Omega)$$

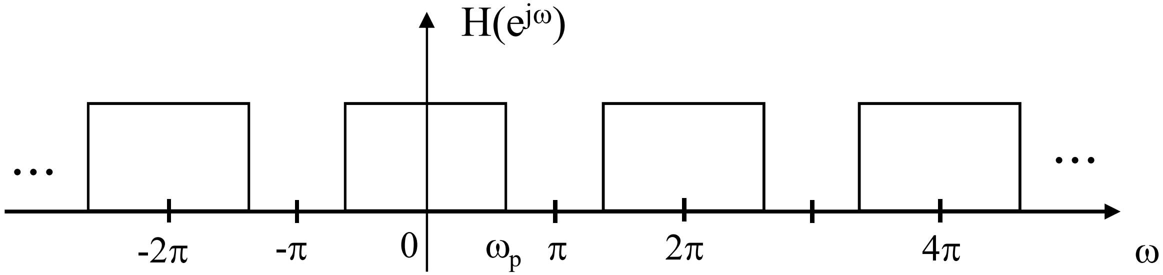
– we may finally conclude that if the discrete-time system is linear and time-invariant, from the input to the output of the system all happens as if there is an analog processing characterized by $H_{eff}(\Omega)$, whose relation to discrete-time processing is:

$$H_{eff}(\Omega) = \begin{cases} H\left(e^{j\Omega T}\right), & |\Omega| < \pi / T \\ 0, & |\Omega| \geq \pi / T \end{cases}$$

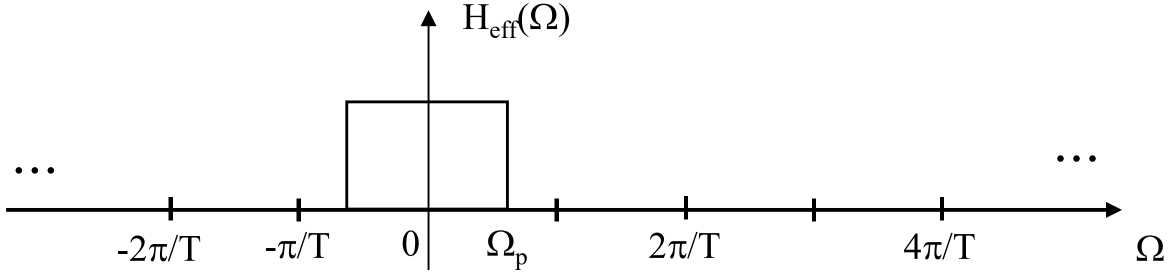
Example: continuous-time low-pass filtering by means of a discrete-time filter

given the filter: $H\left(e^{j\omega}\right) = \begin{cases} 1, & |\omega| < \omega_p \\ 0, & \omega_p < |\omega| \leq \pi \end{cases}$ whose frequency response is

$\omega$-periodic, with period $2\pi$ :



then: $$H_{eff}(\Omega)=\begin{cases}1, & |\Omega T|<\omega_p \\ 0, & |\Omega T|>\omega_p\end{cases}=\begin{cases}1, & |\Omega|<\Omega_p=\omega_p/T \\ 0, & |\Omega|>\Omega_p=\omega_p/T\end{cases}$$




A few reasons justifying that this analog filter implemented in the discrete-time domain may be preferable:

- as the cut-off frequency $\Omega_p=\omega_p/T$ depends on T, using the same system, we may vary the effective analog cut-off frequency (i.e., we have adjustable filters), by acting solely on the sampling frequency (1/T),
- when we need a filter with demanding specifications, involving for example very narrow transition bands, or high stop-band attenuation, or many bands with different gains and attenuations; its realization in the analog domain is difficult, probably very expensive, and highly dependent on the characteristics of the analog components, and in any case it will show a strongly non-linear phase response. Moving that filtering effort to the discrete-time domain eliminates almost completely these inconveniences. A specific case where that is true involves A/D and D/A operations, that require, respectively, “anti-aliasing” and “anti-imaging” filters, both low-pass. The analog filter specifications are ‘alleviated’ (and in certain cases no analog filtering at all is needed) transferring most of the filtering effort to the discrete/digital domain although requiring a significant increase of the sampling frequency. In the first case, (*i.e.* after A/D conversion), decimating digital filters are used and in the second case (*i.e.* before D/A conversion), interpolating digital filters are used. We will return to these topics later on !