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- *The Z-Transform of the auto/cross-correlation*
  - *the Z-Transform of the auto-correlation*
  - *the Z-Transform of the cross-correlation*

The Z-Transform

- consists in a generalization of the Fourier transform for discrete signals
  - allows to represent signals whose Fourier transform does not converge
- is equivalent to the Laplace transform for continuous-time signals
- simplifies the notation in the analysis of problems (*e.g.* interpolation or decimation)

- Definition

$$Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{+\infty} x[n] Z^{-n} \quad , \quad Z = re^{j\omega}$$

where Z is a continuous complex variable, we represent symbolically:

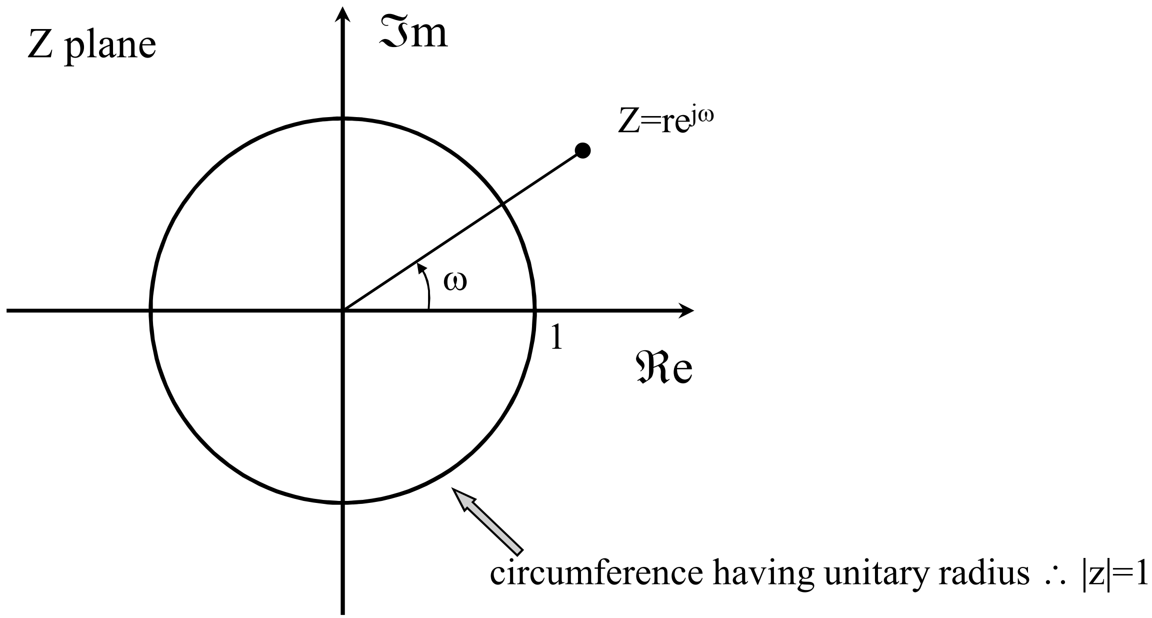
$$x[n] \xleftrightarrow{\;\;Z\;\;} X(z)$$

NOTE: the Z transform of x[n] is the Fourier transform of the signal $x[n]r^{-n}$, such that when r=1, a Z transform reduces to the Fourier transform:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] Z^{-n} = \sum_{n=-\infty}^{+\infty} x[n] \left(re^{j\omega}\right)^{-n} = \sum_{n=-\infty}^{+\infty} \left[x[n] r^{-n}\right] e^{-j\omega n} = F\left\{x[n] r^{-n}\right\}$$

The Z-Transform

- Plane of the Z complex variable



  - **particularity 1:** the Fourier transform corresponds to the evaluation of the Z transform on the unit circumference
  - **particularity 2:** the 2π periodicity that characterizes the representation of a discrete signal in the frequency domain, is intrinsic to the Z plane

The region of convergence of the Z-Transform

- Region of convergence

  – given a discrete sequence x[n], the set of Z values for which the Z transform converges (*i.e.* the infinite summation of power values converges to a finite result) is known as the region of convergence (RoC or RC)

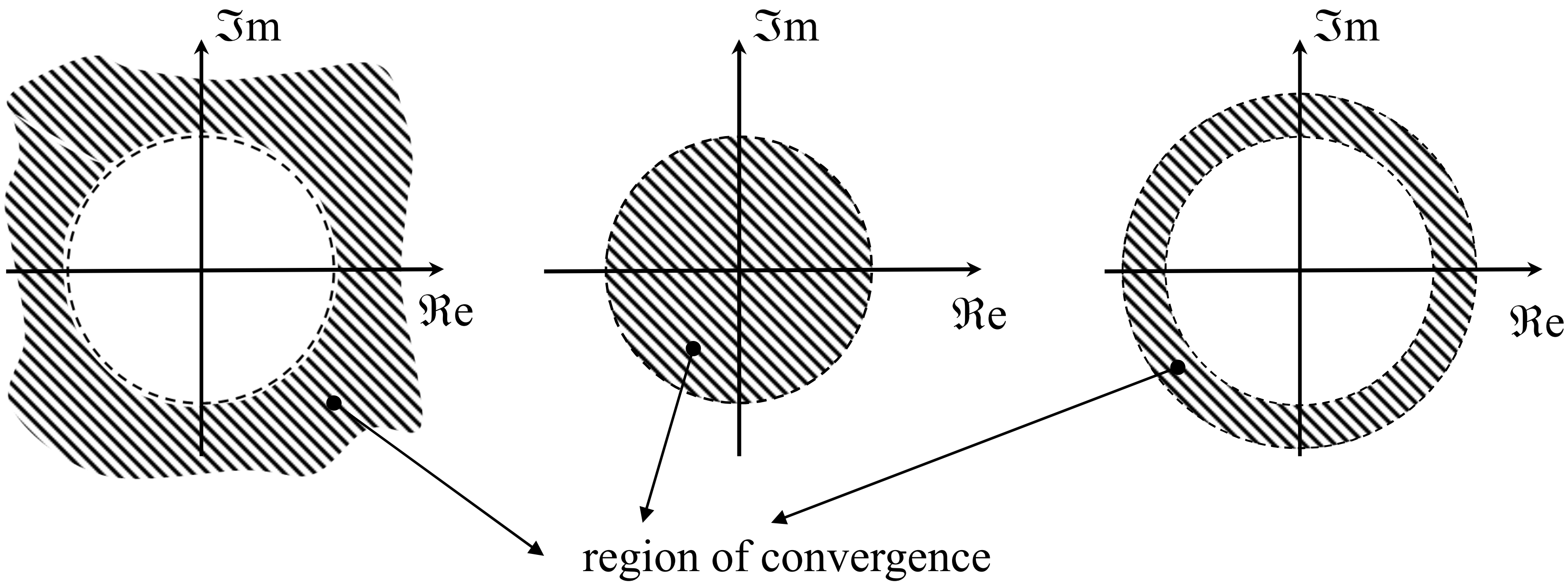
  – the condition to be verified, as in the case of the Fourier transform, is that the sequence of powers of the Z transform is absolutely summable:

$$\sum_{n=-\infty}^{+\infty}\left|x[n]Z^{-n}\right| = \sum_{n=-\infty}^{+\infty}\left|x[n]\right|\left|Z^{-n}\right| = \sum_{n=-\infty}^{+\infty}\left|x[n]\right|\left|r\right|^{-n} < \infty$$

  – from the previous it can be concluded that if Z_1 belongs to the region of convergence, then any Z_2 such that $|Z_1| = |Z_2|$, also belongs to the region of convergence, and hence the RC has always the shape of a ring in the Z plane and centered at the origin of this plane.

The region of convergence of the Z-Transform

– from the previous it results that three possibilities may occur for the RC:



– NOTE: if the region of convergence associated with the Z transform of a discrete-time sequence includes the unit circumference, then it can be concluded that the Fourier transform exists (*i.e.* converges) for that sequence. Inversely, ...

Properties of the RC of the Z-Transform

– the most common and useful way to express mathematically the Z transform of a sequence, using a closed-form expression (*i.e.* using a compact expression), is by means of a rational function:

$$X(z) = \frac{P(z)}{Q(z)}$$

where P(z) and Q(z) are Z polynomials. The finite roots of P(z) are the ZEROES of the Z transform (usually identified by the symbol "o" in the Z plane) and the finite roots of Q(z) are the POLES of the Z transform (*i.e.* they make that X(z) be infinite and they are usually identified by the symbol "x" in the Z plane) . It may however happen that zeroes or poles appear at z=0 (visible) or at $z=\infty$ (not visible).

Taking into consideration the previous ideas, we define the following:

- Properties of the region of convergence (RC)
  1. the RC is a disc or ring in the Z plane and centered at the origin,
  2. the RC is a connected region (*i.e.* it is not the combination of disjoint regions),

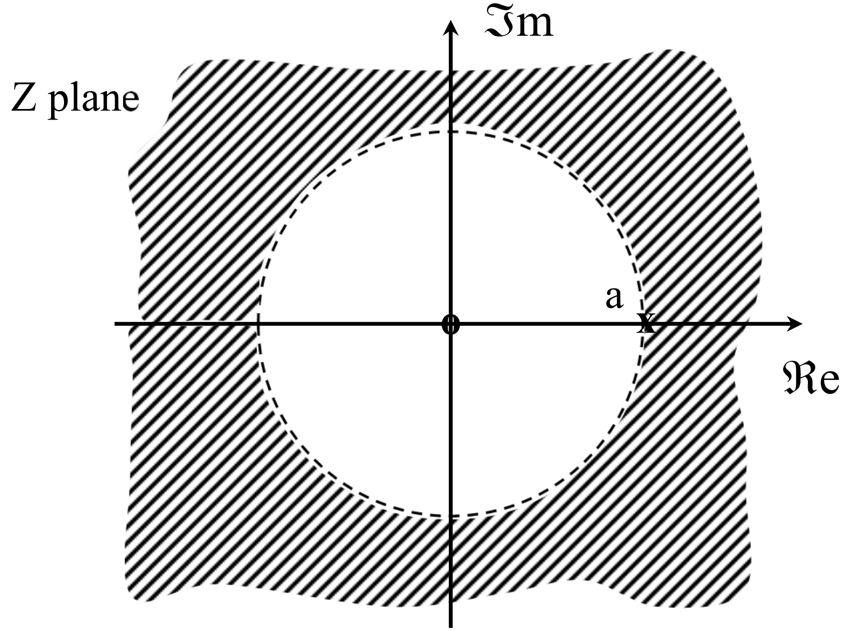
Properties of the RC of the Z-Transform

3. the RC may not contain poles inside,
4. if x[n] is a finite-duration sequence (*i.e.* a sequence that is different from zero for $-\infty < N_1 < n < N_2 < +\infty$) then the RC is the entire Z plane, except possibly for $z=0$ or for $z=\infty$,
5. if x[n] is a <u>right-hand sided sequence</u> (*i.e.* a sequence that is different from zero for $n > N_1 > -\infty$), then the RC extends to the outside of a circumference defined by the finite pole that is more distant from the origin of the Z plane,
6. if x[n] is a <u>left-hand sided sequence</u> (*i.e.* a sequence that is different from zero for $n < N_2 < +\infty$), then the RC extends to the inside of a circumference defined by the finite pole that is closest to the origin of the Z plane,
7. if x[n] is neither right-hand sided nor left-hand sided (*i.e.,* it is a two-sided sequence), then the RC, if it exists, consists in a ring (that may not contain poles inside !), that is bounded by two circumferences defined by two finite poles,
8. the Fourier transform of a sequence x[n] converges absolutely if and only if the RC of its Z transform includes the unit circumference.

example 1

$$x[n] = a^n u[n] \quad \overset{Z}{\longleftrightarrow} \quad X(z) = \sum_{n=0}^{+\infty} a^n Z^{-n} = \sum_{n=0}^{+\infty} \left(aZ^{-1}\right)^n = \frac{1}{1-aZ^{-1}} \quad ,if \quad \left|aZ^{-1}\right| < 1 \therefore |Z| > |a|$$

where RC $\equiv$ $|z| > |a|$



(admitting a is real and positive)

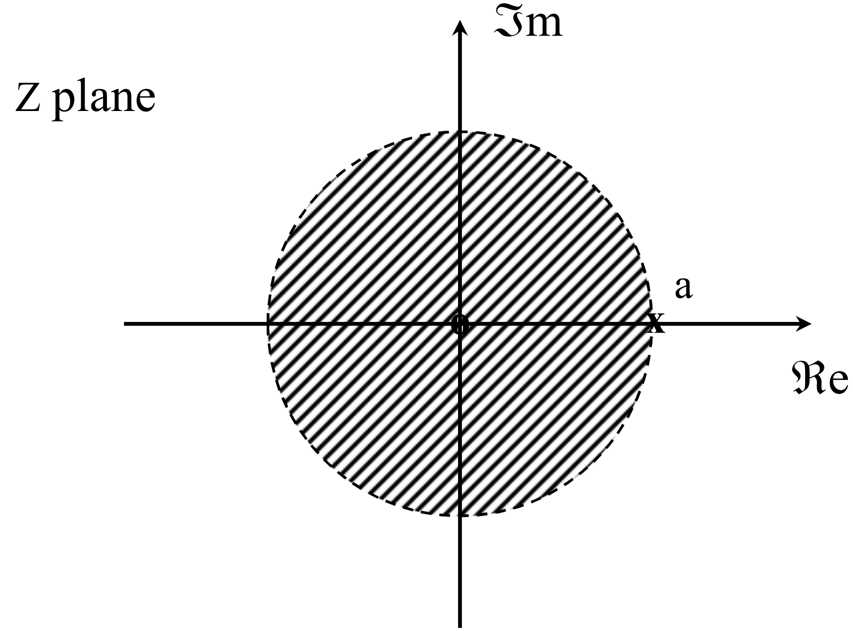
NOTE 1: if |a| < 1, then the Fourier transform of the sequence x[n] exists

NOTE 2: this example includes, as a particular case, the unit step (that is not absolutely summable nor square summable, but whose Fourier transform exists using discontinuous and non-differentiable functions: pulses)

example 2

$$x[n] = -a^n u[-n-1] \quad \overset{Z}{\longleftrightarrow} \quad X(z) = -\sum_{n=-\infty}^{-1} a^n Z^{-n} = -\sum_{n=-\infty}^{-1} \left(aZ^{-1}\right)^n = 1 - \sum_{n=0}^{+\infty} \left(a^{-1}Z\right)^n = 1 - \frac{1}{1-a^{-1}Z} = \frac{1}{1-aZ^{-1}} \quad , if \quad \left|a^{-1}Z\right| < 1 \therefore |Z| < |a|$$

where RC $\equiv$ $|z| < |a|$



(admitting a is real and positive)

NOTE : if |a| > 1, then the Fourier transform of the sequence x[n] exists

– the previous two examples reveal that the Z function defining the poles and the zeroes of the Z transform of a signal is insufficient to characterize it: it is always necessary to specify the associated region of convergence (RC)

– in case x[n] consists of several terms, each one having its own RC, then the combined RC is the intersection among all RCs, *i.e.* the one making simultaneously valid the convergence of the different sums of Z powers, as the following example illustrates.

example 3

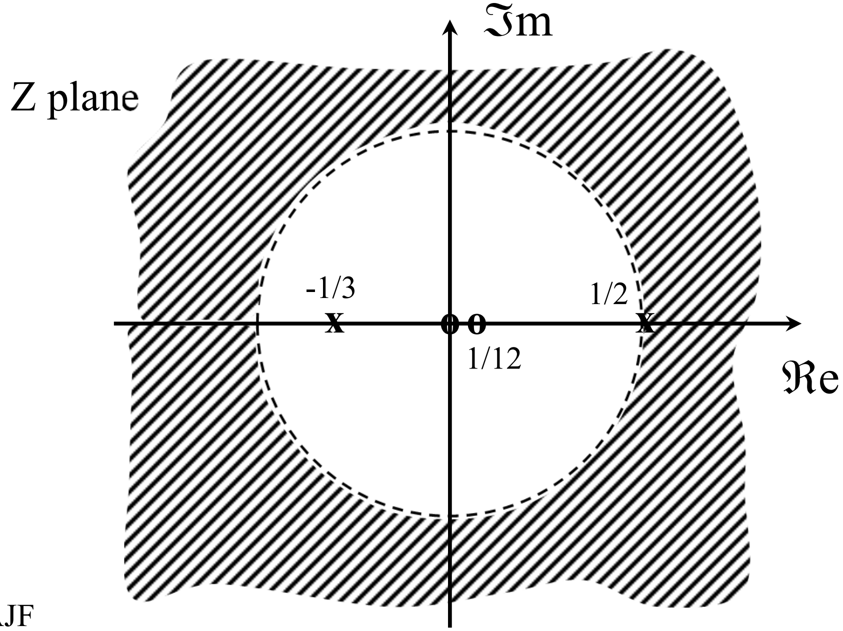
$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \quad \xleftrightarrow{Z} \quad ?$$

as:

$$\left.\begin{array}{l}\boxed{\left(\frac{1}{2}\right)^n u[n]} \\ \boxed{\left(-\frac{1}{3}\right)^n u[n]}\end{array}\right\} \xleftrightarrow{Z} \left\{\begin{array}{l}\boxed{\dfrac{1}{1-\frac{1}{2}Z^{-1}} \quad , |Z| > \frac{1}{2}} \\ \boxed{\dfrac{1}{1+\frac{1}{3}Z^{-1}} \quad , |Z| > \frac{1}{3}}\end{array}\right.$$

and given the linearity of the Z transform:

$$\boxed{\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]} \xleftrightarrow{Z} \boxed{\frac{1}{1-\frac{1}{2}Z^{-1}} + \frac{1}{1+\frac{1}{3}Z^{-1}} = \frac{2\left(1-\frac{1}{12}Z^{-1}\right)}{\left(1-\frac{1}{2}Z^{-1}\right)\left(1+\frac{1}{3}Z^{-1}\right)} = \frac{2Z\left(Z-\frac{1}{12}\right)}{\left(Z-\frac{1}{2}\right)\left(Z+\frac{1}{3}\right)} \quad , |Z| > \frac{1}{2} \cap |Z| > \frac{1}{3} \equiv |Z| > \frac{1}{2}}$$



QUESTION : what would be the RC if $x[n]=(-1/3)^n u[n]-(1/2)^n u[-n-1]$?

example 4

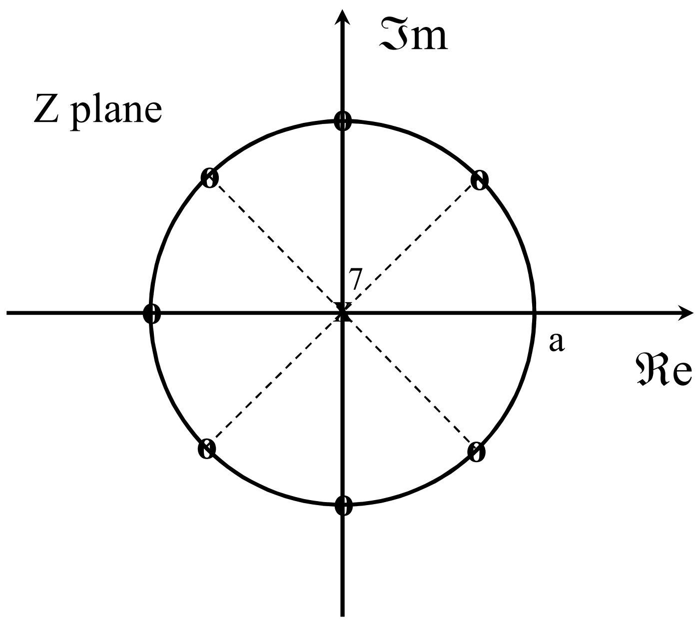
$$x[n]=\begin{cases} a^n & ,0\le n\le N-1 \\ 0 & other \end{cases} \quad \overset{Z}{\longleftrightarrow} \quad X(z)=\sum_{n=0}^{N-1}a^nZ^{-n}=\sum_{n=0}^{N-1}\left(aZ^{-1}\right)^n=\frac{1-\left(aZ^{-1}\right)^N}{1-aZ^{-1}}=\frac{1}{Z^{N-1}}\frac{Z^N-a^N}{Z-a} \quad ,\forall\ Z\setminus\{Z=0\}$$

NOTE 1: the roots of the numerator (zeroes) are given by $Z_k=ae^{jk2\pi/N}$, $0\le k\le N-1$

NOTE 2: the pole at Z=a is cancelled out by the zero at the same location,

NOTE 3: as long as $|aZ^{-1}|$ is finite $\Leftrightarrow$ $|a|<\infty$ and $Z\neq 0$, this case does not imply convergence difficulties and, as a result, the RC is the entire Z plane except Z=0,

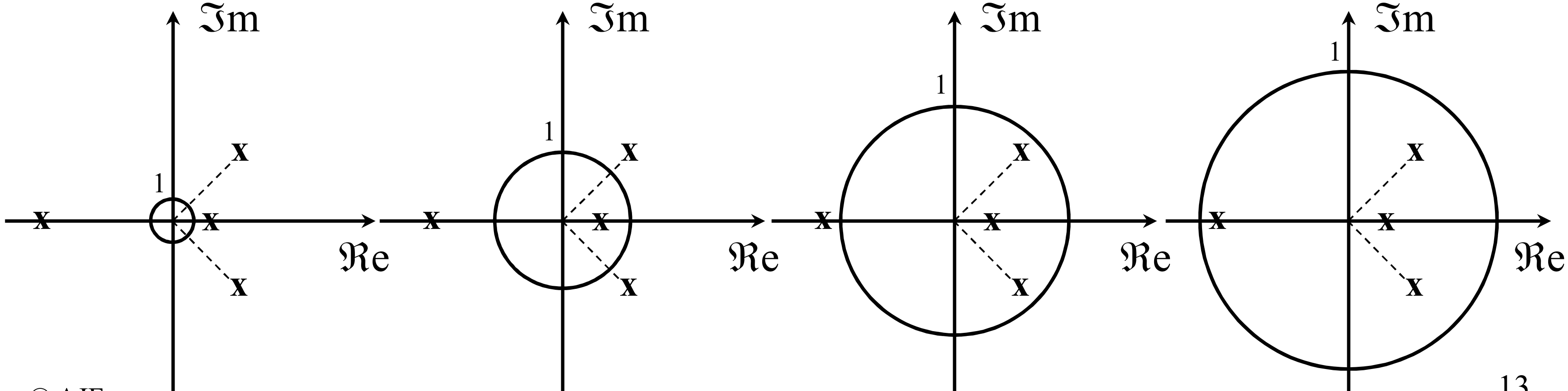
NOTE 4: if N=8, the distribution of poles and zeroes in the Z plane is:



Implications of stability and causality in the RC

1. If a system having impulse response h[n] [whose Z transform is H(z)] <u>is stable</u> (*i.e.* h[n] is absolutely summable and, thus, has a Fourier transform), then the RC associated with H(z) must include the unit circumference
2. If a system having impulse response h[n] <u>is causal</u>, then h[n] is a right-hand sided sequence and the RC associated with its Z transform, H(z), must extend to the outside of a circumference defined by the finite pole that is more far way from the origin of the Z plane.

Question: which of the following zero-pole diagrams may correspond to a discrete system that is simultaneously stable and causal ?



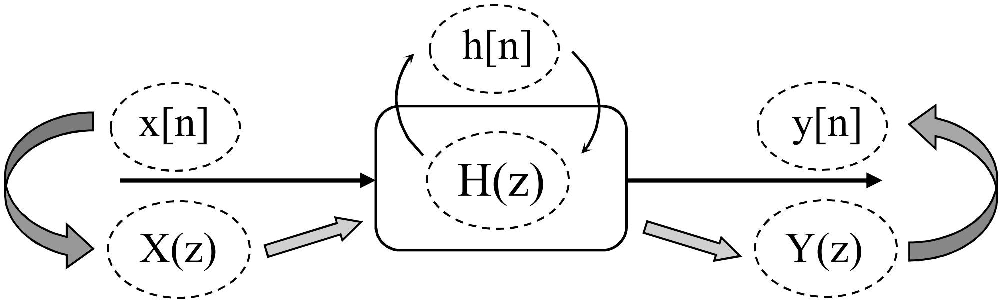
A few important Z-Transform pairs

(useful to evaluate either the direct Z-Transform or the inverse Z-Transform !)

$x[n]$	$X(z)$	RC
$\delta[n]$	**1**	• entire Z plane
$\delta[n-n_0]$	Z^{-n_0}	• entire Z plane except Z=0 (if $n_0 > 0$) or except Z=∞ (if $n_0 < 0$)
$u[n]$	$\frac{1}{1-Z^{-1}}$	• $\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-Z^{-1}}$	• $\lvert z\rvert < 1$
$a^n u[n]$	$\frac{1}{1-aZ^{-1}}$	• $\lvert z\rvert > a$
$-a^n u[-n-1]$	$\frac{1}{1-aZ^{-1}}$	• $\lvert z\rvert < a$
$na^n u[n]$	$\frac{aZ^{-1}}{(1-aZ^{-1})^2}$	• $\lvert z\rvert > a$
$-na^n u[-n-1]$	$\frac{aZ^{-1}}{(1-aZ^{-1})^2}$	• $\lvert z\rvert < a$
$(\cos\omega_0 n)u[n]$	$\frac{1-\cos\omega_0\, Z^{-1}}{1-2\cos\omega_0\, Z^{-1}+Z^{-2}}$	• $\lvert z\rvert > 1$
$(\sin\omega_0 n)u[n]$	$\frac{\sin\omega_0\, Z^{-1}}{1-2\cos\omega_0\, Z^{-1}+Z^{-2}}$	• $\lvert z\rvert > 1$
$(r^n\cos\omega_0 n)u[n]$	$\frac{1-r\cos\omega_0\, Z^{-1}}{1-2r\cos\omega_0\, Z^{-1}+r^2Z^{-2}}$	• $\lvert z\rvert > r$
$(r^n\sin\omega_0 n)u[n]$	$\frac{r\sin\omega_0\, Z^{-1}}{1-2r\cos\omega_0\, Z^{-1}+r^2Z^{-2}}$	• $\lvert z\rvert > r$

The inverse Z-Transform

frequent path in the analysis/project/modification of signals or discrete systems:



the computation of the inverse Z transform is thus necessary and frequent.

- Method 1: by inspection

Involves the identification and direct use of known pairs of the Z transform from a table such as that of the previous slide; in order to take full advantage of this method, it is convenient to decompose the Z function (whose inverse we want to find) as a sum of simple Z functions (*e.g.,* first order functions), such that, for each one, the corresponding Z transform pair is readily identified.

The inverse Z-Transform

- Method 2: partial fraction expansion

if X(z) is expressed as a ratio of Z polynomials:

$$X(z)=\frac{\sum_{k=0}^{M} b_k Z^{-k}}{\sum_{\ell=0}^{N} a_\ell Z^{-\ell}}$$

the number of poles is equal to the number of zeroes and all may be represented in the “finite” Z plane (*i.e.* there are no zeroes or poles at z=∞), hence it is possible to express X(z) as a sum of partial fractions, each one associated to a pole of X(z):

$$X(z)=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}\left(1-c_k Z^{-1}\right)}{\prod_{\ell=1}^{N}\left(1-d_\ell Z^{-1}\right)}$$

where c_k are the non-zero zeroes of X(z) and d_ℓ are the non-zero poles of X(z).

The inverse Z-Transform

If X(z) is presented in an irreducible form, *i.e.* if M<N ***and*** all poles are first order (*i.e.* their multiplicity is 1), then X(z) may be written as:

$$X(z)=\sum_{k=1}^{N}\frac{A_k}{1-d_k Z^{-1}}$$

where the constants A_k are obtained as:

$$A_k=\left(1-d_k Z^{-1}\right)X(z)\Big|_{Z=d_k}$$

Finding the inverse Z transform is now straightforward. That is also the case when M≥N after dividing the numerator by the denominator, the order of the numerator of the remainder must be less than N and X(z) may be expressed as:

$$X(z)=\sum_{\ell=0}^{M-N}B_\ell Z^{-\ell}+\sum_{k=1}^{N}\frac{A_k}{1-d_k Z^{-1}}$$

The inverse Z-Transform

If there are poles whose multiplicity is higher than 1, a more complex approach has to be followed; for example, if a pole exists at d_i whose multiplicity is m then, presuming all other poles are first-order, X(z) may be expressed as:

$$X(z) = \sum_{s=0}^{M-N} B_s Z^{-s} + \sum_{\substack{k=1 \\ k\neq i}}^{N} \frac{A_k}{1-d_k Z^{-1}} + \sum_{\ell=1}^{m} \frac{C_\ell}{\left(1-d_i Z^{-1}\right)^\ell}$$

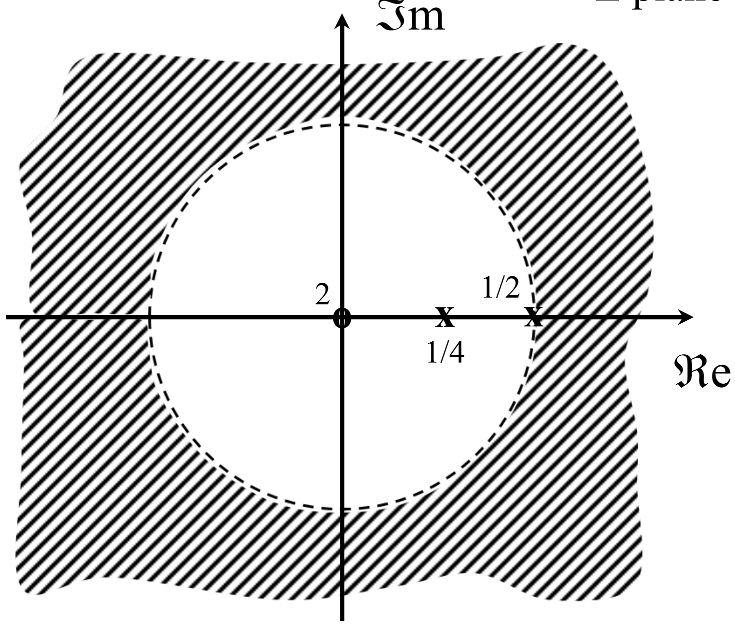
where the constants C_ℓ are obtained as:

$$C_\ell = \frac{1}{(m-\ell)!(-d_i)^{m-\ell}} \left\{ \frac{\partial^{m-\ell}}{\partial w^{m-\ell}} \left[(1-d_i w)^m X(w^{-1}) \right] \right\}_{w=d_i^{-1}}$$

After the decomposition of X(z) as partial fractions, x[n] may be evaluated as the inverse Z transform of each partial fraction and taking into consideration the linearity of the Z transform. The identification of the causal or anti-causal behavior of each partial fraction results by analyzing the regions of convergence.

Example

$$X(z)=\frac{1}{\left(1-\frac{1}{4}Z^{-1}\right)\left(1-\frac{1}{2}Z^{-1}\right)}\quad,\quad |Z|>\frac{1}{2}\quad \overset{Z}{\longleftrightarrow}\quad ?$$



Then: $$X(z)=\frac{A}{1-\frac{1}{4}Z^{-1}}+\frac{B}{1-\frac{1}{2}Z^{-1}}$$

where: $$A=\left(1-\frac{1}{4}Z^{-1}\right)X(z)\Big|_{Z=\frac{1}{4}}=-1$$ and $$B=\left(1-\frac{1}{2}Z^{-1}\right)X(z)\Big|_{Z=\frac{1}{2}}=2$$

resulting: $$X(z)=\frac{2}{1-\frac{1}{2}Z^{-1}}-\frac{1}{1-\frac{1}{4}Z^{-1}}$$ and by analyzing the region of convergence, one concludes that the two poles are associated to right-hand sided sequences, which gives rise to:

$$x[n]=2\left(\frac{1}{2}\right)^n u[n]-\left(\frac{1}{4}\right)^n u[n]$$

not to forget !

- Method 3: contour integral

Taking advantage of the Cauchy integral theorem which states that:

$$\frac{1}{2\pi j}\oint_C Z^{k-1-\ell}\, dZ = \begin{cases} 1 & ,\ k=\ell \\ 0 & ,\ k\neq \ell \end{cases}$$

(particular case: ℓ=0) where C is a counter-clockwise contour that includes the origin of the Z plane, one may conclude [Oppenheim, 1975] that it is possible to find x[n] using the contour integral:

$$x[n] = \frac{1}{2\pi j}\oint_C X(z) Z^{n-1}\, dZ$$

where C is a counter-clockwise contour inside the RC [Sanjit Mitra, 2006].

The inverse Z-Transform

The advantage of this formulation is that for rational functions, it may be conveniently replaced by the computation of the residue theorem:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) Z^{n-1} dZ = \sum \left[\text{residues of } \; X(z) Z^{n-1}\text{, at the poles inside C} \right]$$

where the residue for a pole at $Z=Z_0$ and having multiplicity m is given by:

$$\text{Residue}\left[X(z) Z^{n-1} \quad at \quad Z = Z_0 \right] = \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left[(Z - Z_0)^m X(z) Z^{n-1} \right]_{z=z_0}$$

NOTE 1: in case of a single pole at $Z=Z_0$ the corresponding residue is:

$$\text{Residue}\left[X(z) Z^{n-1} \quad at \quad Z = Z_0 \right] = (Z - Z_0) X(z) Z^{n-1} \Big|_{z=z_0}$$

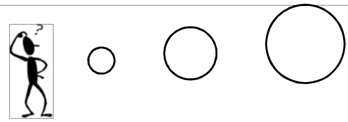
NOTE 2: the utilization of this method for n<0 may be problematic since a pole at z=0 and having multiplicity > 1 may appear. As an alternative, it may be preferable to use other methods.

The inverse Z-Transform

- EXAMPLE $$X(z)=\frac{1}{\left(1-\frac{1}{4}Z^{-1}\right)\left(1-\frac{1}{2}Z^{-1}\right)}\quad,\quad |Z|>\frac{1}{2}\quad \xleftrightarrow{\;Z\;}\quad ?$$

we have: $$X(z)Z^{n-1}=\frac{Z^{n-1}}{\left(1-\frac{1}{4}Z^{-1}\right)\left(1-\frac{1}{2}Z^{-1}\right)}=\frac{Z^{n+1}}{\left(Z-\frac{1}{4}\right)\left(Z-\frac{1}{2}\right)}$$

and thus:



$$\text{Residue at } z=\frac{1}{4}: \quad \left(Z-\frac{1}{4}\right)X(z)Z^{n-1}\Big|_{z=\frac{1}{4}}=\frac{\left(\frac{1}{4}\right)^{n+1}}{\frac{1}{4}-\frac{1}{2}}=-\left(\frac{1}{4}\right)^{n},\quad n\geq 0$$

$$\text{Residue at } z=\frac{1}{2}: \quad \left(Z-\frac{1}{2}\right)X(z)Z^{n-1}\Big|_{z=\frac{1}{2}}=\frac{\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}-\frac{1}{4}}=2\left(\frac{1}{2}\right)^{n},\quad n\geq 0$$

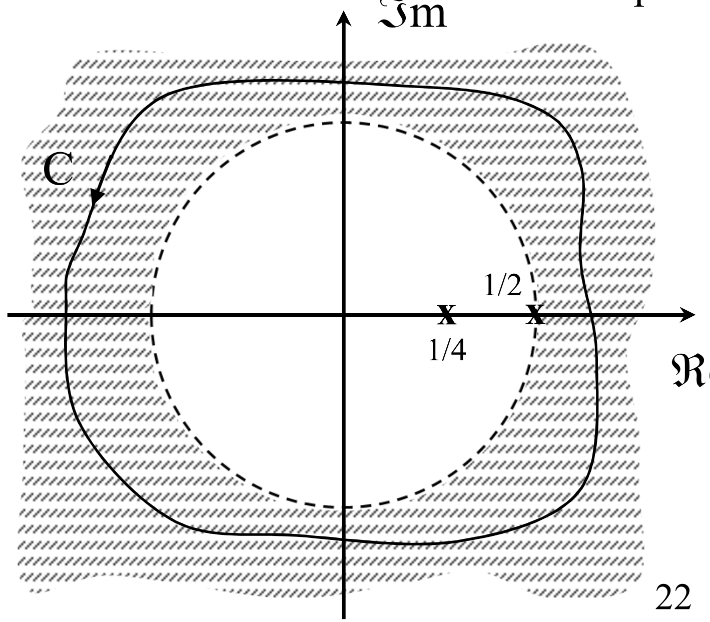
from which we conclude:

$$x[n]=\left[2\left(\frac{1}{2}\right)^{n}-\left(\frac{1}{4}\right)^{n}\right]u[n]$$

QUESTION: how to proceed if, presuming causality, $X(z)Z^{n-1}$ was:

$$\frac{Z^{n-1}}{\left(Z-\frac{1}{4}\right)\left(Z-\frac{1}{2}\right)}\quad ?$$

Z plane



Properties of the Z-Transform

– Properties are very useful in the analysis and project of discrete-time signals and systems (allowing for example a direct connection between a difference equation describing a system and the Z transform of its impulse response).

taking: $\boxed{x[n]} \xleftrightarrow{Z} \boxed{X(z) \quad , \quad with \quad RC = R_X \equiv \quad r_E < |Z| < r_D}$

and also: $\left.\begin{matrix}\boxed{x_1[n]} \\ \\ \boxed{x_2[n]}\end{matrix}\right\} \xleftrightarrow{Z} \left\{\begin{matrix}\boxed{X_1(z) \quad , \quad with \quad RC = R_{X1}} \\ \\ \boxed{X_2(z) \quad , \quad with \quad RC = R_{X2}}\end{matrix}\right.$

we have:

- Linearity

$$\boxed{ax_1[n] + bx_2[n]} \xleftrightarrow{Z} \boxed{aX_1(z) + bX_2(z) \quad , \quad with \quad RC = R_{X1} \cap R_{X2}}$$

NOTE: a linear combination may give rise to a pole-zero cancellation and hence the final RC may be larger than R_{X1} and R_{X2} , for example:

$\boxed{x[n] = a^n u[n] - a^n u[n-N]}$ but the final RC is |Z|>0 .

$\Uparrow \quad \Uparrow$

$\boxed{RC_1 \equiv |Z| > |a|}$ $\boxed{RC_2 \equiv |Z| > |a|}$

- Displacement in n

$$x[n-n_0] \xleftrightarrow{\;Z\;} Z^{-n_0}X(z) \quad , \quad RC = R_X \text{ (except for possible insertion/ deletion of } z=0 \text{ or } z=\infty)$$

example:

$$X(z) = \frac{1}{Z-\frac{1}{4}} = \frac{Z^{-1}}{1-\frac{1}{4}Z^{-1}} = -4 + \frac{4}{1-\frac{1}{4}Z^{-1}} \xleftrightarrow{\;Z\;} -4\delta[n] + 4\left(\frac{1}{4}\right)^n u[n] = x[n]$$

$$RC \equiv |Z| > \frac{1}{4}$$

but it is also, in a more direct and simplified way:

$$RC \equiv |Z| > \frac{1}{4}$$

$$\left.\begin{array}{l} X_1(z) = \dfrac{1}{1-\frac{1}{4}Z^{-1}} \\ X(z) = Z^{-1}X_1(z) \end{array}\right\} \xleftrightarrow{\;Z\;} \left\{\begin{array}{l} \left(\frac{1}{4}\right)^n u[n] = x_1[n] \\ x[n] = x_1[n-1] = \left(\frac{1}{4}\right)^{n-1} u[n-1] = -4\delta[n] + 4\left(\frac{1}{4}\right)^n u[n] \end{array}\right.$$

Properties of the Z-Transform

- Multiplication by a complex exponential

$$Z_0^n x[n] \xleftrightarrow{Z} X\left(\frac{Z}{Z_0}\right) \quad , \quad RC = |Z_0| R_X \equiv \quad |Z_0| r_E < |Z| < |Z_0| r_D$$

the implication of this operation is to scale all poles and zeroes of X(z) by $|Z_0|$ in the radial direction in case Z_0 is a positive real number, or to rotate all poles and zeroes of X(z) by ω_0 radians, relatively to the origin, in case $Z_0=e^{j\omega 0}$. This last case corresponds to the modulation property in the Fourier domain (in case the Fourier transform exists):

$$e^{j\omega_0 n} x[n] \xleftrightarrow{F} X\left[e^{j(\omega-\omega_0)}\right]$$

example:

$$\left.\begin{array}{r} u[n] \\ \\ x[n] = r^n \cos(\omega_0 n) u[n] \end{array}\right\} \xleftrightarrow{Z} \left\{\begin{array}{l} \dfrac{1}{1-Z^{-1}} \quad , \quad RC = |Z| > 1 \\ \\ \quad ? \end{array}\right.$$

solution:

as $$x[n] = r^n \cos(\omega_0 n) u[n] = \frac{1}{2}\left(re^{j\omega_0}\right)^n u[n] + \frac{1}{2}\left(re^{-j\omega_0}\right)^n u[n]$$

we have:

$$\left.\begin{array}{l} \frac{1}{2}\left(re^{j\omega_0}\right)^n u[n] \\ \frac{1}{2}\left(re^{-j\omega_0}\right)^n u[n] \end{array}\right\} \stackrel{Z}{\longleftrightarrow} \left\{\begin{array}{l} \dfrac{1/2}{1 - re^{j\omega_0} Z^{-1}} \quad , \quad RC = |Z| > \mathrm{r} \\ \dfrac{1/2}{1 - re^{-j\omega_0} Z^{-1}} \quad , \quad RC = |Z| > \mathrm{r} \end{array}\right.$$

and hence: $$X(z) = \frac{1/2}{1 - re^{j\omega_0} Z^{-1}} + \frac{1/2}{1 - re^{-j\omega_0} Z^{-1}} = \frac{1 - r\cos\omega_0 Z^{-1}}{1 - 2r\cos\omega_0 Z^{-1} + r^2 Z^{-2}} \quad , \quad RC = |Z| > \mathrm{r}$$

- Differentiation of X(z)

$$nx[n] \stackrel{Z}{\longleftrightarrow} -Z\frac{dX(z)}{dZ} \quad , \quad RC = R_X$$

example:

$$X(z) = \log\left(1 + aZ^{-1}\right) \quad , \quad |Z| > \mathrm{a} \stackrel{Z}{\longleftrightarrow} \; ?$$

Properties of the Z -Transform

solution:

in this case we have: $\frac{dX(z)}{dZ} = -\frac{aZ^{-2}}{1+aZ^{-1}}$

and thus: $nx[n] \overset{Z}{\longleftrightarrow} -Z\frac{dX(z)}{dZ} = \frac{aZ^{-1}}{1+aZ^{-1}} \overset{Z}{\longleftrightarrow} a(-a)^{n-1}u[n-1]$

and finally: $x[n] = \frac{a(-a)^{n-1}u[n-1]}{n}$

- Conjugation

$$x^*[n] \overset{Z}{\longleftrightarrow} X^*(z^*) \quad , \quad RC = R_X$$

- Inversion in n

$$\left.\begin{matrix} x[-n] \\ x^*[-n] \end{matrix}\right\} \overset{Z}{\longleftrightarrow} \left\{\begin{matrix} X(1/z) \quad , \quad RC = 1/R_X \equiv \quad 1/r_D < |Z| < 1/r_E \\ X^*(1/z^*) \quad , \quad RC = 1/R_X \end{matrix}\right.$$

Properties of the Z-Transform

example:

$$\left.\begin{array}{r} a^n u[n] \\ x[n] = a^{-n} u[-n] \end{array}\right\} \overset{Z}{\longleftrightarrow} \left\{\begin{array}{l} \dfrac{1}{1-aZ^{-1}} \quad , \quad |Z| > a \\ X(z) = \dfrac{1}{1-aZ} = \dfrac{-a^{-1}Z^{-1}}{1-a^{-1}Z^{-1}} \quad , \quad RC = |Z| < 1/a \end{array}\right.$$

- Convolution

$$x_1[n] * x_2[n] \overset{Z}{\longleftrightarrow} X_1(z)X_2(z) \quad , \quad RC = R_{X1} \cap R_{X2}$$

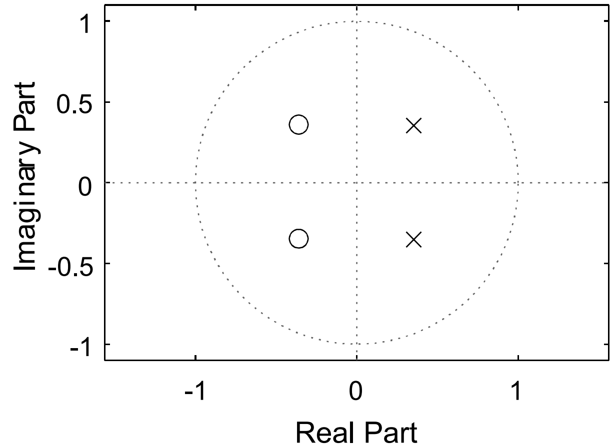
NOTE: as a result of this operation, *pole-zero cancellation* may occur between the zeroes and poles of the Z function, such that the final RC may be larger than R_{X1} and R_{X2}

– The convolution property is fundamental in the sense that the Z transform of the output of an LTI system is given by the product between the Z transform of the input and the Z transform of the impulse response of the system, commonly known as the <u>transfer function</u>

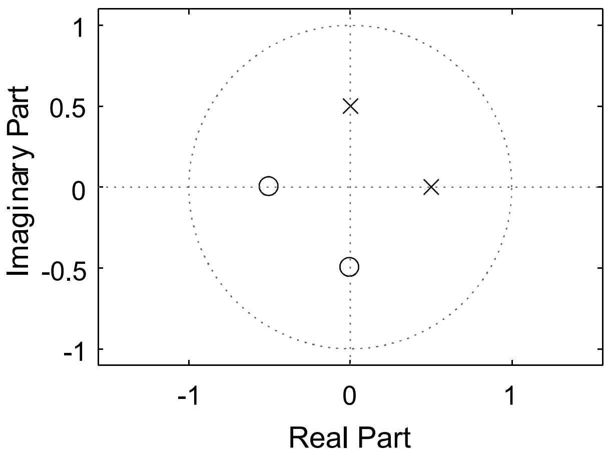
example of the multiplication by a complex exponential property:

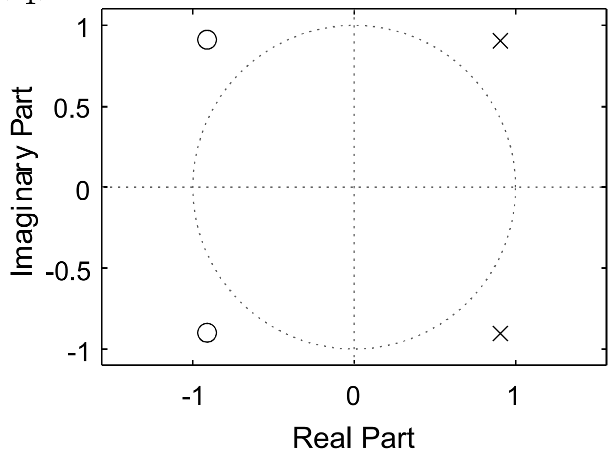
let us consider a second-order Z-Transform, $H(z)$, whose zero-pole distribution in the Z-plane is as follows:

```
zero0 = 0.5*exp(1j*3*pi/4);
zero1 = 0.5*exp(1j*5*pi/4);

pole0 = 0.5*exp(1j*pi/4);
pole1 = 0.5*exp(1j*7*pi/4);

b=poly([zero0 zero1]);
a=poly([pole0 pole1]);

zplane(b,a)
```



let us now consider the transformations $H\left(\frac{z}{z_0}\right)$, and $H\left(\frac{z}{z_1}\right)$, where $z_0 = 2.56$ and $z_1 = e^{j\pi/4}$





Questions: which plot corresponds to the z_0 transformation ? and to the z_1 transformation ? why ?

Do all the plots correspond to real-valued discrete-time sequences ?

Properties of the Z-Transform

- Multiplication

$$x_1[n]\cdot x_2^*[n] \xleftrightarrow{Z} \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{Z^*}{v^*}\right)v^{-1}\,dv \quad , \quad RC = R_{X1}\cdot R_{X2}$$

where $R_{X1} \equiv \quad r_{E1} < |Z| < r_{D1}\ , R_{X2} \equiv \quad r_{E2} < |Z| < r_{D2} \quad , \quad R_{X1}\cdot R_{X1} \equiv \quad r_{E1}\cdot r_{E2} < |Z| < r_{D1}\cdot r_{D2}$

NOTE: C is a closed counter-clockwise contour in the area of intersection between the convergence region of $X_1(v)$ and that of $X_2(Z/v)$. The multiplication property is also known as the modulation theorem or the complex convolution theorem.

- Generalization of the Parseval theorem to the Z domain

As:

$$w[n] = x_1[n]\cdot x_2^*[n] \xleftrightarrow{Z} W(Z) = \sum_{n=-\infty}^{+\infty} x_1[n]\cdot x_2^*[n]Z^{-n}$$

then:

$$\sum_{n=-\infty}^{+\infty} x_1[n]\cdot x_2^*[n] = W(1) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}\,dv$$

or, changing the variable v into z: $$\sum_{n=-\infty}^{+\infty} x_1[n]\cdot x_2^*[n] = \frac{1}{2\pi j}\oint_C X_1(Z)X_2^*\left(\frac{1}{Z^*}\right)Z^{-1}\,dZ$$

If both $X_1(Z)$ and $X_2^*(1/Z^*)$ include the unit circumference in their convergence regions, it is possible to use it as the closed C contour and hence $z=e^{j\omega}$, which leads to:

$$\sum_{n=-\infty}^{+\infty} x_1[n]\cdot x_2^*[n] = \frac{1}{2\pi}\int_{\omega=-\pi}^{\pi} X_1(e^{j\omega})X_2^*\left(e^{j\omega}\right)d\omega$$

As a particular case, the energy of a signal may be evaluated in the Z domain:

$$\sum_{n=-\infty}^{+\infty} \left|x[n]\right|^2 = \frac{1}{2\pi}\int_{\omega=-\pi}^{\pi}\left|X(e^{j\omega})\right|^2 d\omega = \frac{1}{2\pi j}\oint_C X(Z)X^*\left(\frac{1}{Z^*}\right)Z^{-1}\,dZ$$

- Initial value theorem

is x[n] is causal (*i.e.*, unilateral Z transform), then: $$x[0] = \lim_{z\to\infty} X(z)$$

(gain of the transfer function)

- Final value theorem

if x[n] is causal (*i.e.*, unilateral Z transform), such as that X(z) has all its poles inside the unit circumference, except possibly for a first-order pole at Z=1, then:

(gain at low frequencies)

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(1-z^{-1})X(z)$$

- the Z-Transform of the auto-correlation

the auto-correlation is defined as (in this discussion, we admit energy signals)

$$r_x[\ell] = x[\ell] * x^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]x^*[k-\ell]$$

considering the Z-Transform properties

$$x[\ell] \overset{Z}{\longleftrightarrow} X(z), \quad RoC = R_x \equiv r_E < |z| < r_D$$

$$x^*[\ell] \overset{Z}{\longleftrightarrow} X^*(z^*), \quad RoC = R_x$$

$$x[-\ell] \overset{Z}{\longleftrightarrow} X(z^{-1}), \quad RoC = 1/R_x \equiv 1/r_D < |z| < 1/r_E$$

$$x^*[-\ell] \overset{Z}{\longleftrightarrow} X^*(1/z^*), \quad RoC = 1/R_x$$

Then

$$r_x[\ell] = x[\ell] * x^*[-\ell] \overset{Z}{\longleftrightarrow} R_x(z) = X(z) \cdot X^*(1/z^*), \; RoC = R_x \cap 1/R_x$$

where $R_x(z) = X(z) \cdot X^*(1/z^*)$ is called the energy spectrum

- the Z-Transform of the auto-correlation (cont.)
  - the Wiener-Khintchine Theorem: the auto-correlation and the energy spectrum form a Z-Transform pair

$$r_x[\ell] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad R_x(z) = X(z) \cdot \; X^*(1/z^*)$$

thus,

$$r_x[\ell] = \frac{1}{2\pi j} \oint_C \; R_x(z)\, Z^{\ell-1} dz$$

and, in particular, the energy of the signal can be found using

$$E = r_x[0] = \sum_{k=-\infty}^{+\infty} |x[k]|^2 = \frac{1}{2\pi j} \oint_C \; X(z) \cdot \; X^*(1/z^*)\, Z^{-1} dz$$

which reflects the Parseval Theorem in the Z-domain

- the Z-Transform of the cross-correlation

the cross-correlation is defined as (we admit energy signals)

$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]\, y^*[k-\ell]$$

considering the Z-Transform properties

$$
\begin{aligned}
&x[\ell] &&\xleftrightarrow{Z} && X(z), && RoC = R_x\\
&y[\ell] &&\xleftrightarrow{Z} && Y(z), && RoC = R_y\\
&y^*[\ell] &&\xleftrightarrow{Z} && Y^*(z^*), && RoC = R_y\\
&y[-\ell] &&\xleftrightarrow{Z} && Y(z^{-1}), && RoC = 1/R_y\\
&y^*[-\ell] &&\xleftrightarrow{Z} && Y^*(1/z^*), && RoC = 1/R_y
\end{aligned}
$$

then

$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] \xleftrightarrow{\mathcal{F}} R_{xy}(z) = X(z) \cdot Y^*(1/z^*),\;\; RoC = R_x \,\cap\, 1/R_y$$