Time and Frequency Domains

z-transform

Why Introduce the *z*-Transform?

The design and analysis of digital filters is aided by an understanding of the *z*-transform.

What Is the *z*-Transform?

- The z-transform is the discrete-time equivalent of the Laplace transform.
- The Laplace transform is a generalization of the continuous-time Fourier transform.
- The z-transform is a generalization of the discrete-time Fourier transform.

Uses of the Laplace Transform

- Solution of continuous-time, linear differential equations
- Representation of continuous-time, linear time invariant systems as s-transfer functions
- Complex variable s may be viewed as an operator, representing differentiation with respect to time.

Uses of the *z*-Transform

- Solution of discrete-time, linear difference equations
- Representation of discrete-time, linear time invariant systems as z-transfer functions
- Complex variable z may be viewed as an operator representing a shift of one sample in a sequence.

Definition of the *z*-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

X(*z*) is a continuous function of complex variable *z.*

x(*n*) is a discrete-time sequence. It is a signal, but, if considered to be an impulse response, it may represent, or characterize, a system.

z-Transform and Fourier Transform

The z-transform is concerned with discrete-time signals (sequences).

The corresponding form of Fourier analysis is the discrete-time Fourier transform (DTFT).

The DTFT is defined as:

$$X(\hat{\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\hat{\omega}}$$

z-Transform and DTFT

Expressing complex variable z in polar form, i.e., $z = re^{j\omega}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)(re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (x(n)r^{-n})e^{-j\omega n}$$

The z-transform of $x(n)$ is equivalent to the DTFT of $x(n)r^{-n}$.

If r = 1, then the z-transform of $x(n)$ is equivalent to the DTFT of $x(n)$.

Recall that $|z| = r$.

z-Transform of an Exponential Function

$$x(n) = a^n u(n)$$

From the definition of the *z*-transform

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

Comparing this with the Taylor series approximation

$$\sum_{n=0}^{\infty} p^n = \frac{1}{1-p} \quad \text{for} \quad |p| < 1$$

We can write $X(z) = \dfrac{1}{1-az^{-1}} = \dfrac{z}{z-a}$ for $|z| > |a|$

There is a range of values of *z* , *a region of convergence,* for which *X(z)* is *absolutely summable*, i.e. , $|X(z)| < \infty$.

z-Transform of an Exponential Function

$$x(n) = -a^n u(-n-1)$$

From the definition of the z-transform

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

Letting $m = -n$

$$X(z) = -\sum_{m=1}^{\infty} a^{-m} z^m = 1 - \sum_{m=0}^{\infty} (za^{-1})^m = 1 - \frac{1}{1 - a^{-1}z} = \frac{z}{z-a}$$

for $|z| < |a|$

z-Transform of an Exponential Function

Two different sequences have exactly the same algebraic expression for their z-transform.

$$x(n) = -a^n u(-n-1) \quad \Rightarrow \quad X(z) = \frac{z}{z-a} \quad \text{for} \quad |z| < |a|$$

$$x(n) = a^n u(n) \quad \Rightarrow \quad X(z) = \frac{z}{z-a} \quad \text{for} \quad |z| > |a|$$

The algebraic expression for a *z*-transform alone does not specify a unique sequence.
The condition part of the *z*-transform must be taken into account.

The conditions on the values of *z* define regions of convergence (ROCs).

Poles and Zeros of a *z*-Transform

The poles of a *z*-transform are the values of *z* for which

$$X(z) \to \infty$$

The zeros of a *z*-transform are the values of *z* for which

$$X(z) = 0$$

For rational *X(z)*, i.e., when *X(z)* is a ratio of polynomials in *z*, you can find/plot poles and zeros.

Regions of Convergence

A region of convergence (ROC) specifies the values of the complex variable z for which $|X(z)| < \infty$.

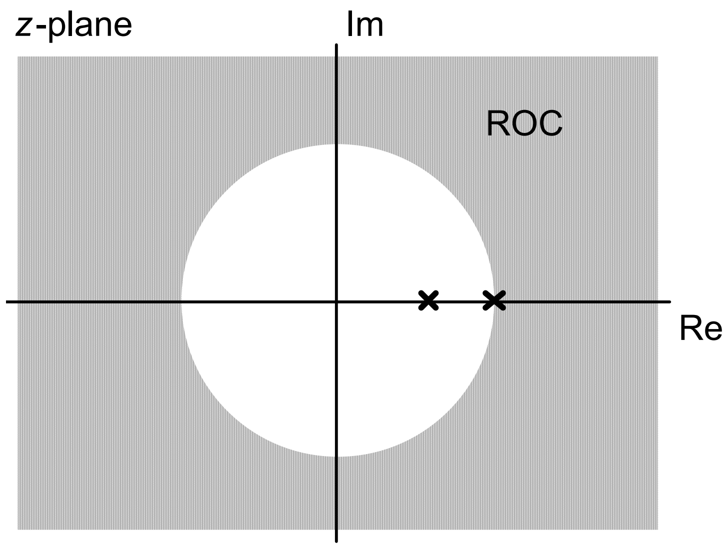
Its graphical interpretation is as an area in the (complex) z-plane.

ROC Properties

- The ROC of $X(z)$ does not contain any poles of $X(z)$.
- ROC boundaries depend on $|z|$, and hence are circles in the z-plane (centered on the origin).
- A ROC is a single connected region in the z-plane.
- If $x(n)$ is finite, then the ROC is the entire z-plane except, possibly, for 0 or infinity.
- Stable systems (for which a discrete-time Fourier transform exists) have ROCs that contain the unit circle.

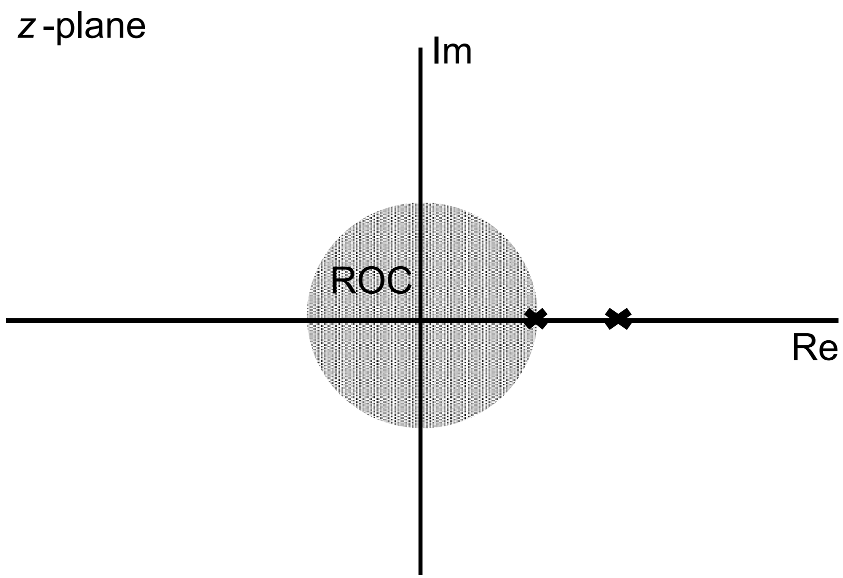
ROC Properties

If $x(n)$ is right-sided, then the ROC extends outward from a circle containing the outermost pole of $X(z)$.



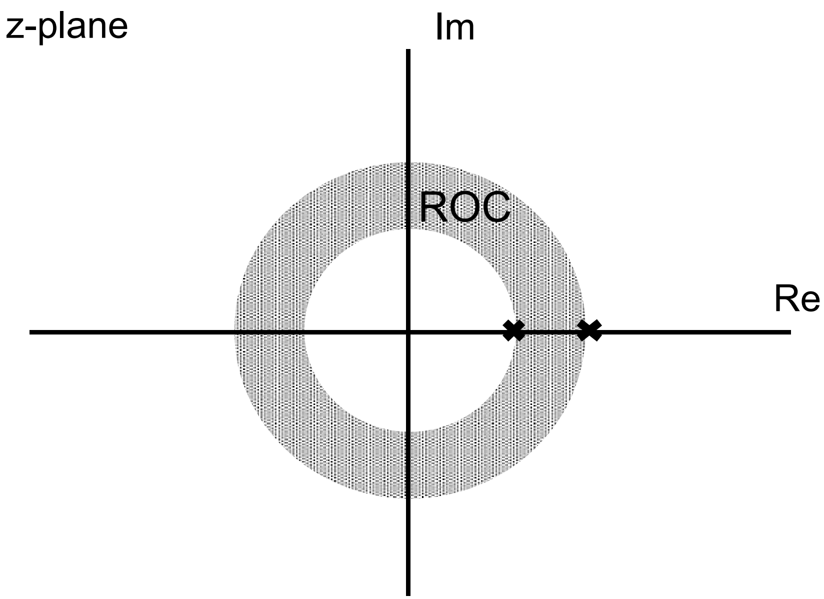
ROC Properties

If $x(n)$ is left-sided, then the ROC is inside a circle that does not contain the innermost pole of $X(z)$.



ROC Properties

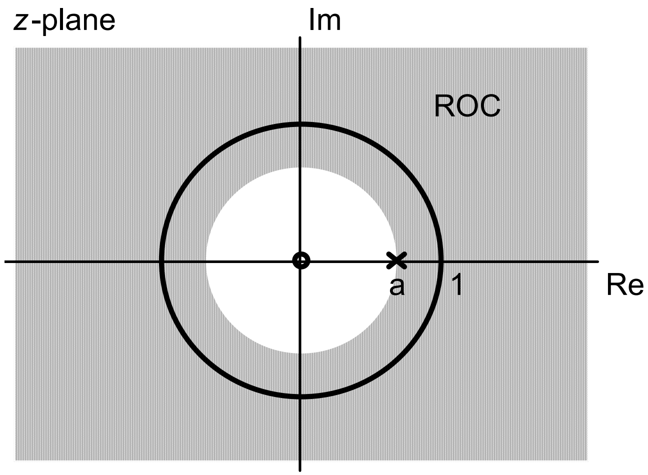
If $x(n)$ is two-sided, then the ROC is a ring bounded by circles inside the outermost pole of $X(z)$ and outside the innermost pole of $X(z)$.



ROCs and Poles and Zeros

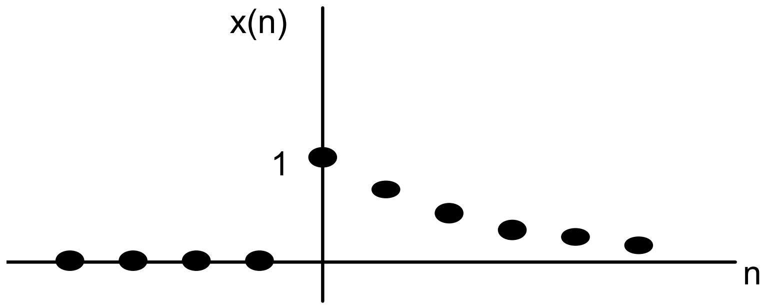
- ROCs may be deduced from poles and zeros of $X(z)$.
- More than one ROC may be compatible with a particular set of poles and zeros.
- Different ROCs correspond to different (impulse) response sequences $x(n)$ with different causality and stability characteristics.
- If ROC extends outward to infinity, then the system is causal.
- Since most physical systems, including the digital filters we will implement, are causal, we will be concerned mainly with such ROCs and with right-sided sequences (signals) starting at $n=0$ and for which the one-sided or unilateral z-transform may be used.

ROC and the Unit Circle



$$X(z) = \frac{z}{z-a}$$

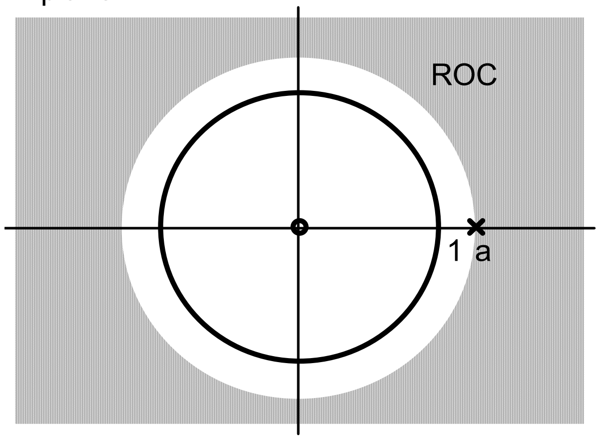
$$a < 1$$



$$x(n) = a^n u(n)$$

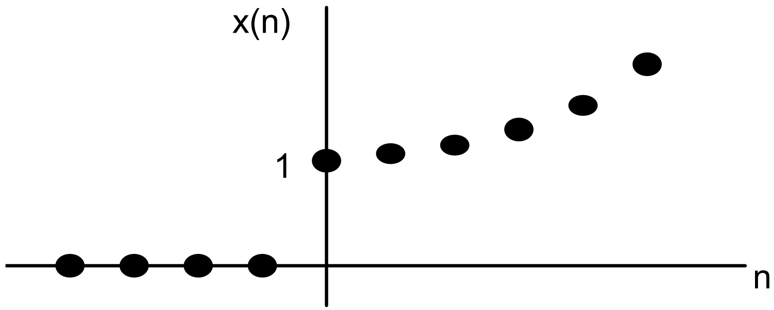
Causal and stable impulse response

ROC and the Unit Circle



$$X(z) = \frac{z}{z-a}$$

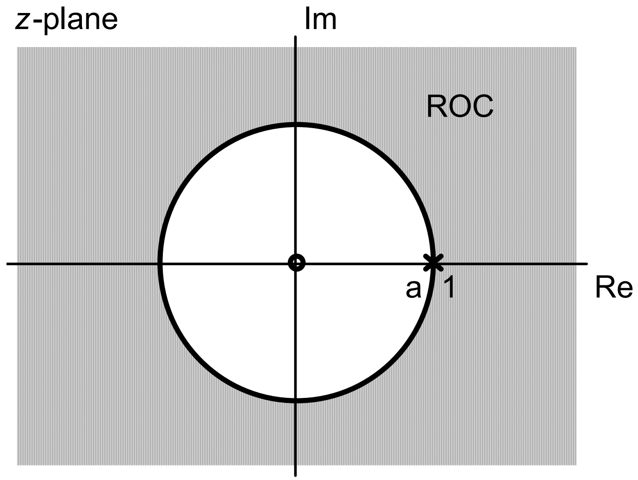
$$a > 1$$



$$x(n) = a^n u(n)$$

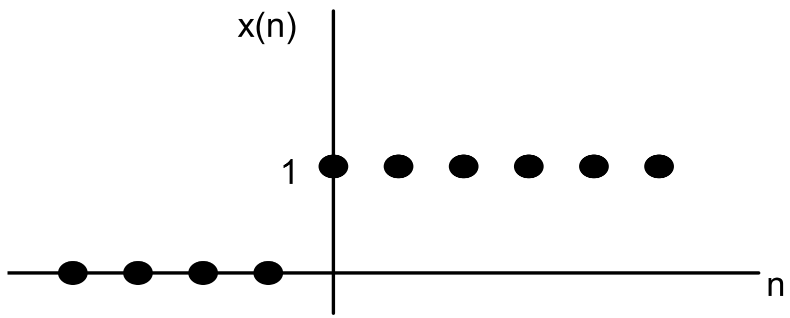
Causal and unstable impulse response

ROC and the Unit Circle



$$X(z) = \frac{z}{z-a}$$

$$a = 1$$



$$x(n) = a^n u(n)$$

Causal and unstable impulse response

Properties of the *z*-Transform

Linearity

If $Z\{x(n)\} = X(z)$ and $Z\{y(n)\} = Y(z)$

then $Z\{ax(n) + by(n)\} = aX(z) + bY(z)$

Properties of the *z*-Transform

Time Delay or Shift

If $$Z\{x(n)\} = X(z)$$

then $$Z\{x(n-m)\} = z^{-m} X(z)$$

Quantity z^{-m} in the *z*-domain corresponds to a shift of *m* sampling instants in the time domain.

Properties of the z-Transform

Convolution

The forced output $y(n)$ of an LTI system having impulse response $h(n)$ and input $x(n)$ is given by the convolution sum

$$y(n) = \sum_{m=0}^{\infty} h(m)x(n-m)$$

Taking the z-transform of this

$$Y(z) = Z\left\{\sum_{m=0}^{\infty} h(m)x(n-m)\right\}$$

$$= \sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty} h(m)x(n-m)\right] z^{-n}$$

Properties of the *z*-Transform

Changing the order of summation

$$Y(z) = \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} h(m)x(n-m) \right] z^{-n}$$

$$= \sum_{m=0}^{\infty} h(m) \left[\sum_{n=0}^{\infty} x(n-m) \right] z^{-n}$$

Letting *l* = *n*-*m*

$$Y(z) = \sum_{m=0}^{\infty} h(m) \sum_{l=0}^{\infty} x(l) z^{-l} z^{-m}$$

$$= \sum_{m=0}^{\infty} h(m) z^{-m} \sum_{l=0}^{\infty} x(l) z^{-l}$$

$$= H(z)X(z)$$

Properties of the *z*-Transform

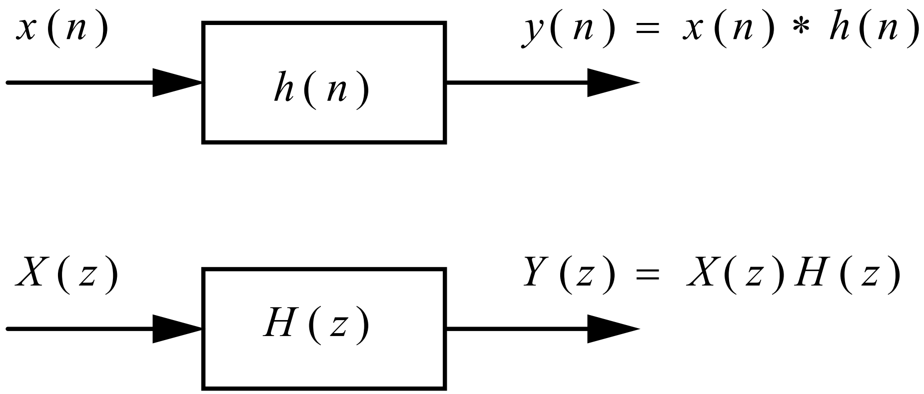
In other words

$$Z\left\{h(n) * x(n)\right\} = H(z)X(z)$$

The *z*-transform of the linear convolution of sequences *h*(*n*) and *x*(*n*) is equal to the product of their *z*-transforms *H*(*z*) and *X*(*z*).

Properties of the *z*-Transform

The convolution property leads to the concept of the *z*-transfer function where $h(n)$ is the impulse response of a system.



Inverse z-Transform

In theory, the inverse z-transform is found by contour integration but, in practice, usually found by partial fraction expansion and the use of z-transform tables.

$x(n)$	$X(z)$	ROC
$d(n)$	1	all z
$a^n u(n)$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
$-a^n u(-n-1)$	$\frac{z}{z-a}$	$\|z\| < \|a\|$

z-transform tables should include ROCs.