PDSiTPclass18MAR2021

Class # 5

$y[n] = x[n] + 2x[n-1] + x[n-2] + \frac{1}{2}y[n-1] + \frac{1}{2}y[n$

a)
$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}, \text{ causal}$$

$$= \frac{\text{"b"}}{\text{"a"}}$$

b = [1 2 1];
a = [1 -0.5 -0.5];

b)

c) $$H(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}} = \frac{1+2z^{-1}+z^{-2}}{(1-z^{-1})(1+\frac{1}{2}z^{-1})}, \; RoC \equiv |z|>$$

$$z^2 - \frac{1}{2}z - \frac{1}{2} = 0 \quad \therefore \quad z = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4(-\frac{1}{2})}}{2} = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4}+2}}{2} = \begin{cases} 1 \\ -\frac{1}{2} \end{cases}$$

$$\sqrt{\frac{1}{4}+2} = \sqrt{\frac{9}{4}} = \frac{3}{2}$$

$$\cancel{= \frac{A}{1-z^{-1}} + \frac{B}{1+\frac{1}{2}z^{-1}}}$$

$$\begin{array}{ccc|l} z^{-2} & 2z^{-1} & 1 & -\frac{1}{2}z^{-2} \quad -\frac{1}{2}z^{-1} \quad 1 \\ -z^{-2} & -z^{-1} & +2 & -2 \\ \hline 0 & z^{-1} & 3 & \end{array}$$

$$= -2 + \frac{3 + z^{-1}}{\ldots} = -2 + \frac{A}{\ldots} + \frac{B}{\ldots}$$

$= F(z)$

$$(1-z^{-1})\left(1+\tfrac{1}{2}z^{-1}\right) \qquad 1-z^{-1} \qquad 1+\tfrac{1}{2}z^{-1}$$

$$A = (1-z^{-1})\,F(z)\Big|_{z=1} = \frac{3+z^{-1}}{1+\frac{1}{2}z^{-1}}\Big|_{z=1} = \frac{4}{1+\frac{1}{2}} = \frac{8}{3}$$

$$B = \left(1+\tfrac{1}{2}z^{-1}\right)F(z)\Big|_{z=-\frac{1}{2}} = \frac{3+z^{-1}}{1-z^{-1}}\Big|_{z=-\frac{1}{2}} = \frac{3-2}{1-(-2)} = \frac{1}{3}$$

$$= -2 + \frac{8/3}{1-z^{-1}} + \frac{1/3}{1+\frac{1}{2}z^{-1}} = H(z)$$

$RoC \equiv |z|>1$ $\qquad$ $RoC \equiv |z| > \frac{1}{2}$

$$h[n] = -2\,\delta[n] + \frac{8}{3}\,u[n] + \frac{1}{3}\left(-\frac{1}{2}\right)^n u[n]$$

d) $$h[n] = \frac{1}{2\pi j}\oint H(z)\, z^{n-1}\, dz = \sum_{\text{residues}} H(z)\, z^{n-1}$$

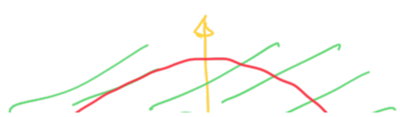
$$H(z)\, z^{n-1} = \frac{1+2z^{-1}+z^{-2}}{(1-z^{-1})\left(1+\frac{1}{2}z^{-1}\right)}\, z^{n-1}$$

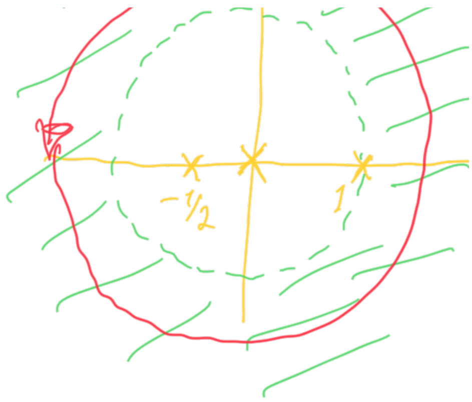
$RoC \equiv |z|>1$

$n \geq 0$

$\boxed{n=0}$ $\therefore$ we have 3 poles: $z=1,\ z=-\frac{1}{2},\ z=0$

$$H(z)\, z^{-1} = \frac{z^2+2z+1}{z(z-1)\left(z+\frac{1}{2}\right)}$$





$$h[0] = \text{residue } H(z)\, z^{-1} z \Big|_{z=0}$$
$$+ \text{ residue } H(z)\, z^{-1} (z-1) \Big|_{z=1}$$
$$+ \text{ residue } H(z)\, z^{-1} \left(z+\tfrac{1}{2}\right) \Big|_{z=-\frac{1}{2}}$$

$$= \frac{z^2+2z+1}{(z-1)\left(z+\frac{1}{2}\right)}\Bigg|_{z=0} + \frac{z^2+2z+1}{z\left(z+\frac{1}{2}\right)}\Bigg|_{z=1} + \frac{z^2+2z+1}{z(z-1)}\Bigg|_{z=-\frac{1}{2}}$$

$$= \frac{1}{(-1)\left(\frac{1}{2}\right)} + \frac{4}{1+\frac{1}{2}} + \frac{\frac{1}{4}-1+1}{\left(-\frac{1}{2}\right)\underbrace{\left(-\frac{1}{2}-1\right)}_{=-\frac{3}{2}}} =$$

$$= -2 + \frac{8}{3} + \frac{1/4}{+\frac{3}{4}} = -2 + \frac{8}{3} + \frac{1}{3} = 1$$

$\boxed{n \geq 1}$

$H(z)\, z^{n-1}$ $\therefore$ just two residues

$$h[n] = \left(1-z^{-1}\right) H(z)\, z^{n-1}\Big|_{z=1} + \left(1+\tfrac{1}{2}z^{-1}\right) H(z)\, z^{n-1}\Big|_{z=-\frac{1}{2}}$$

$$= \frac{z^2+2z+1}{z+\frac{1}{2}}\, z^{n-1}\Bigg|_{z=1} + \frac{z^2+2z+1}{z-1}\, z^{n-1}\Bigg|_{z=-\frac{1}{2}}$$

$$= \frac{4}{3} + \frac{\frac{1}{4}+2\left(-\frac{1}{2}\right)+1}{-1 \quad -1} \left(-\frac{1}{2}\right)^{n-1}$$

$2 \qquad z^{-1}$

$$= \frac{8}{3} + \frac{1/4}{-3/2}\left(-\frac{1}{2}\right)^{n-1} = \frac{8}{3} - \frac{1}{6}\left(-\frac{1}{2}\right)^{n-1}$$

$n \geq 1$

$$h[n] = \delta[n] + \left(\frac{8}{3} - \frac{1}{6}\left(-\frac{1}{2}\right)^{n-1}\right)u[n-1]$$

$\boxed{2}$

$x[n]$: number of new students at year n

$y[n]$: total # of " " " "

success rate : 70% $\quad \therefore$ fail rate is 30%

a) $y[n] = x[n] + 0.3\,y[n-1]$, causal

$$H(z) = \frac{1}{1-0.3z^{-1}}, \quad |z| > 0.3$$

$$x[n] = 100\,u[n] \quad \therefore \quad X(z) = \frac{100}{1-z^{-1}}, \quad \text{RoC} \equiv |z| > 1$$

$$Y(z) = H(z)X(z) = \frac{100}{(1-z^{-1})(1-0.3z^{-1})}, \quad |z| > 1$$

$$Y(z) = \frac{A}{1-z^{-1}} + \frac{B}{1-0.3z^{-1}}$$

$$A = (1-z^{-1})\,Y(z)\Big|_{z=1}$$

$$= \frac{100}{1-\frac{3}{10}z^{-1}}\Big|_{z=1}$$

$B = (1-0.3z^{-1})\,Y(z)|$

$$B = (1-0.3z^{-1})\,Y(z)\Big|_{z=\frac{3}{10}} = \frac{100}{1-z^{-1}}\Big|_{z=\frac{3}{10}} = \frac{100}{1-\frac{10}{3}} = -\frac{300}{7}$$

$$= \frac{100}{7/10} = \frac{1000}{7}$$

$$= \frac{1000/7}{1-z^{-1}} - \frac{300/7}{1-0.3z^{-1}}$$

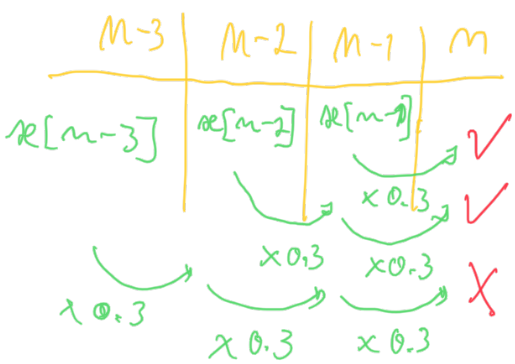
$$|z|>1 \qquad\qquad |z|>0.3$$

$$y[n] = \frac{1000}{7}u[n] - \frac{300}{7}(0.3)^n u[n]$$

$$\lim_{n\to+\infty} y[n] = \frac{1000}{7} \simeq 143$$

c) if a student fails for the 3rd time, he/she is not allowed to enrol again

$$y[n] = x[n] + 0.3\,y[n-1] - 0.3^3 x[n-3]$$



$\therefore\ 0.3^3\, x[n-3]$

new students

$$H(z) = \frac{1-0.3^3 z^{-3}}{1-0.3z^{-1}} = \frac{1-(0.3z^{-1})^3}{1-0.3z^{-1}} = 1 + 0.3z^{-1} + (0.3z^{-1}$$

students who failed once

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$$X(z) = \frac{100}{1-z^{-1}}, \quad |z|>1$$

$$Y(z) = H(z)\,X(z)$$

$$= \left(1 + 0.3\,z^{-1} + 0.3^2\,z^{-2}\right)\frac{100}{1-z^{-1}}$$

$$= \underbrace{\frac{100}{1-z^{-1}}}_{|z|>1} + \underbrace{\frac{30\,z^{-1}}{1-z^{-1}}}_{|z|>1} + \underbrace{\frac{9\,z^{-2}}{1-z^{-1}}}_{|z|>1}$$

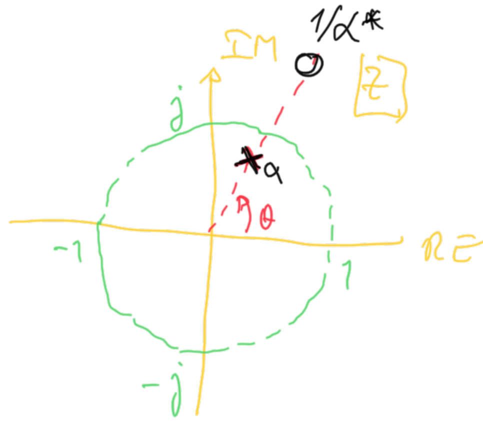
$$y[n] = 100\,u[n] + 30\,u[n-1] + 9\,u[n-2]$$

$$\lim_{n\to+\infty} y[n] = 100 + 30 + 9 = 139$$

$\boxed{5.}$

pole: α, $|\alpha|<1$

zero: $1/\alpha^*$



$\alpha = r e^{j\theta}$

a) $$H(z) = \frac{z - \frac{1}{\alpha^*}}{z-\alpha}$$

all-pass: $|H(e^{j\omega})|^2 = 1 = H(e^{j\omega})\,H^*(e^{j\omega})$

$= H(z)\cdot H^*(1/z^*)\,|$

$$= H(z)\, H^*(1/z^*)\Big|_{z=e^{j\omega}}$$

$$h[n] * h^*[-n] \xrightarrow[\mathcal{F}]{\mathcal{Z}} = 1$$

$$= \delta[n]$$

$$H(z)\,H^*(1/z^*) = \frac{z - 1/\alpha^*}{z-\alpha} \cdot \left(\frac{\frac{1}{z^*} - \frac{1}{\alpha^*}}{\frac{1}{z^*}-\alpha}\right)^*$$

$$= \frac{z-\frac{1}{\alpha^*}}{z-\alpha}\cdot\frac{\frac{1}{z} - \frac{1}{\alpha}}{\frac{1}{z} - \alpha^*}$$

$$= \frac{z-\frac{1}{\alpha^*}}{z-\alpha}\cdot\frac{1-\frac{z}{\alpha}}{1-z\alpha^*}$$

$$= \frac{z - \frac{z^2}{\alpha} - \frac{1}{\alpha^*} + \frac{z}{\alpha\alpha^*}}{z - z^2\alpha^* - \alpha + z\alpha\alpha^*}$$

$$= \frac{1}{\alpha\alpha^*}\,\frac{\alpha\alpha^* z - \alpha^* z^2 - \alpha + z}{z - z^2\alpha^* - \alpha + z\alpha\alpha}$$

$$= \frac{1}{\alpha\alpha^*} = \frac{1}{|\alpha|^2}$$

to continue...