PDSi30MAR2021

Lecture # 14

$$x[n] \xrightarrow{DTFT} X(e^{j\omega}) \xrightarrow{\text{sampling}} Y[k] = X(e^{j\omega})\Big|_{\omega = k\frac{2\pi}{N}}$$

[aperiodic] 2π-periodic [N-periodic]

$k = 0, 1, \ldots, N-$

$$Y[k] \xrightarrow{DFT^{-1}} y[n] \overset{?}{=} f\{x[n]\}$$

[N-periodic] is N-periodic, but how if in this context we assume that $x[n]$ is aperiodic ?

DFT DEFINITION

analysis:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

(N-periodic)

$k = 0, 1, \ldots, N-1$

, $\boxed{W_a^b = e^{-j2\pi\frac{b}{a}}}$

$$W_N^{kn} = e^{-j\frac{2\pi}{N}kn}$$

synthesis:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-kn}$$

(N-periodic)

$n = 0, 1, \ldots, N-1$

$$x[n] \overset{DFT}{\longleftrightarrow} X[k]$$

[N-periodic] [N-periodic]

if $x[n]$ is aperiodic:

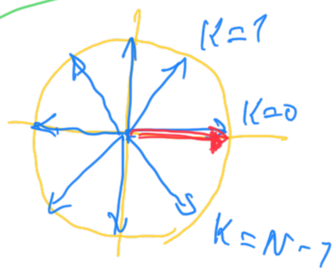
$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} x[l]\, e^{-j\omega l}$$

$$\underset{\text{[N-periodic]}}{Y[k]} = X(e^{j\omega})\Big|_{\omega = k\frac{2\pi}{N}} = \sum_{l=-\infty}^{+\infty} x[l]\, e^{-jk\frac{2\pi}{N}l}$$

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k]\, W_N^{-kn} \qquad = e^{+j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} \sum_{l=-\infty}^{+\infty} x[l]\, e^{-jk\frac{2\pi}{N}l}\, e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{l=-\infty}^{+\infty} x[l] \sum_{k=0}^{N-1} e^{-jk\frac{2\pi}{N}(l-n)}$$



$$= \begin{cases} N, & \text{if } l-n = mN, \quad m \in \mathbb{Z} \\ 0, & \text{other cases} \end{cases}$$

$$= \frac{1}{N}\sum_{m=-\infty}^{+\infty} x[n+mN]\, N$$

$l-n = mN$

$l = n + mN$

$+\infty$

$$= \sum_{m=-\infty} x[n+mN]$$

$$= \cdots + x[n+N] + x[n] + x[n-N] + \cdots$$

conclusion: $y[n]$ is an N-periodic sequence that is obtained by combining (i.e. adding) an infinite number of replicas of the aperiodic sequence $x[n]$; these replicas consist of shifted versions of $x[n]$ when that shift is a multiple integer of N.

conclusion: sampling the signal representation in one domain turns the signal representation in the other domain periodic.