PDSI06MAY2021

Lecture #22

- The DFT as a filter bank
- The frequency response of each filter of the DFT
- Introduction to spectrum estimation

FILTERING:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

"filter"

$$\boxed{= \sum_{\ell=-\infty}^{+\infty} x[\ell]\, h[n-\ell]}$$

DFT:

$$X[k] = \boxed{\sum_{\ell=0}^{N-1} x[\ell]\, e^{-j\frac{2\pi}{N}k\ell}}$$

"frequency" "time"

$$x[n] \longrightarrow \boxed{\text{DFT}} \longrightarrow X[k]$$

$$w[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$X_k[N-1] = \sum_{\ell=(N-1)-(N-1)}^{N-1} x[\ell]\, e^{j\frac{2\pi}{N}((N-1)-(N-1)-\ell)k}$$

n

$$X_k[n] = \sum^{n} x[\ell]\, e^{-j\frac{2\pi}{N}(N-1)k}\, e^{j\frac{2\pi}{N}(n-\ell)k}$$

$$X_k[n] = \sum_{l = n-(N-1)} x[l] \; e^{\dots} \dots$$

$$w[n-l] = \begin{cases} 1, & 0 \leq n-l \leq N-1 \\ 0, & \text{other} \end{cases}$$

$$0 \leq n-l \leq N-1$$

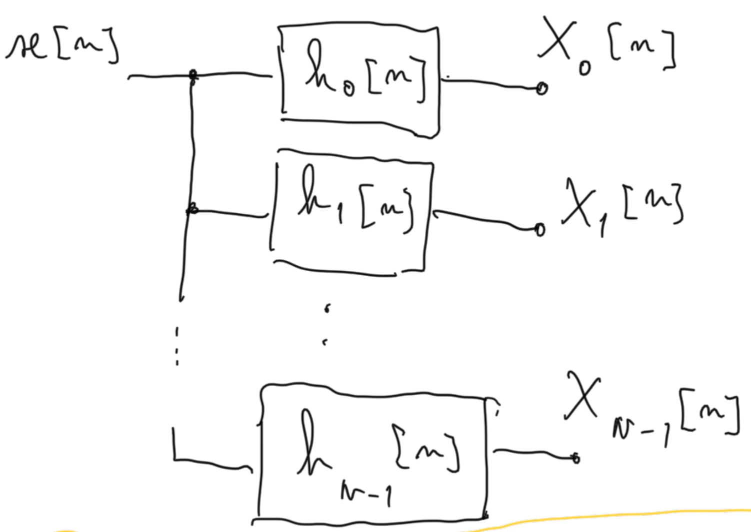
$$0 \geq -n+l \geq -(N-1)$$

$$n-(N-1) \leq l \leq n$$

$$= \sum_{l=-\infty}^{+\infty} x[l] \; \underbrace{e^{-j\frac{2\pi}{N}(N-1)k} \; w[n-l] \; e^{j\frac{2\pi}{N}(n-l)k}}_{= h_k[n-l]}$$

$$= \sum_{l=-\infty}^{+\infty} x[l] \; h_k[n-l]$$

$$k = 0, 1, \dots, N-1$$



TAKE-HOME MESSAGE:

the DFT is a bank of N parallel filters

$$\omega_k = \frac{2\pi}{N} k$$

$$h_k[n] = w[n] \; e^{-j\frac{2\pi}{N}(N-1)k} \; e^{j\frac{2\pi}{N} n k}$$

question:

question.

how do the filters look like?

$$H_k(e^{j\omega}) \triangleq \sum_{n=-\infty}^{+\infty} h_k[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} w[n]\, e^{-j\frac{2\pi}{N}(N-1)k}\, e^{j\omega_k n}\, e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(N-1)k}\, e^{-j(\omega-\omega_k)n}$$

$$= e^{-j\frac{2\pi}{N}(N-1)k} \sum_{n=0}^{N-1} \left(e^{-j(\omega-\omega_k)}\right)^n$$

$$= \frac{1-e^{-j(\omega-\omega_k)N}}{1-e^{-j(\omega-\omega_k)}}$$

$$= \frac{e^{-j(\omega-\omega_k)\frac{N}{2}}}{e^{-j(\omega-\omega_k)\frac{1}{2}}} \quad \frac{\cancel{2j}\sin(\omega-\omega_k)\frac{N}{2}}{\cancel{2j}\sin(\omega-\omega_k)\frac{1}{2}}$$

$$= e^{-j\frac{2\pi}{N}(N-1)k}\, e^{-j(\omega-\omega_k)\frac{N-1}{2}} \quad \frac{\sin(\omega-\omega_k)\frac{N}{2}}{\sin(\omega-\omega_k)\frac{1}{2}}$$

$$|H_k(e^{j\omega})| = \left|\sin\left((\omega-\omega_k)\frac{N}{2}\right)\right|$$

$$|H_k(e^{j\omega})| = \left| \frac{\text{[illegible]}}{\sin\left((\omega-\omega_k)\frac{1}{2}\right)} \right| \qquad k = 0, 1, \ldots, N$$

$$\omega_k \Big|_{k=0} = \frac{2\pi}{N} k = 0$$

$$|H_0(e^{j\omega})| = \left| \frac{\sin\left(\omega \frac{N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \right|$$

i.e., the first DFT filter is a periodic SINC function

$$\omega \frac{N}{2} = m\pi \quad \therefore \ \omega = m\frac{2\pi}{N}$$

question: how about the other filters, $k = 1, 2, \ldots, N-1$?

(NEXT CLASS ...)