# PDSI 06/11 MAY2021

Lecture #22 and #23

- The DFT as a filter bank
- The frequency response of each filter of the DFT
- Introduction to spectrum estimation

FILTERING:


$$y[n] = h[n] * x[n] = \sum_{\ell=-\infty}^{+\infty} x[\ell]\, h[n-\ell]$$

DFT:
$$X[k] = \sum_{\ell=0}^{N-1} x[\ell]\, e^{-j\frac{2\pi}{N}k\ell}$$

"frequency" ($k$) "time" ($\ell$)


$$w[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$




$$X_k[N-1] = \sum_{\ell=(N-1)-(N-1)}^{N-1} x[\ell]\, e^{j\frac{2\pi}{N}((N-1)-(N-1)-\ell)k}$$

($N-1 \to n$)

$$\ldots \sum^{n} x[\ell]\, e^{-j\frac{2\pi}{N}(N-1)k}\, e^{j\frac{2\pi}{N}(n-\ell)k}$$

$$X_k[n] = \sum_{l=n-(N-1)} x[l] \, e \ldots$$

$$w[n-l] = \begin{cases} 1, & 0 \leq n-l \leq N-1 \\ 0, & \text{other} \end{cases}$$

$$0 \leq n-l \leq N-1$$

$$0 \geq -n+l \geq -(N-1)$$

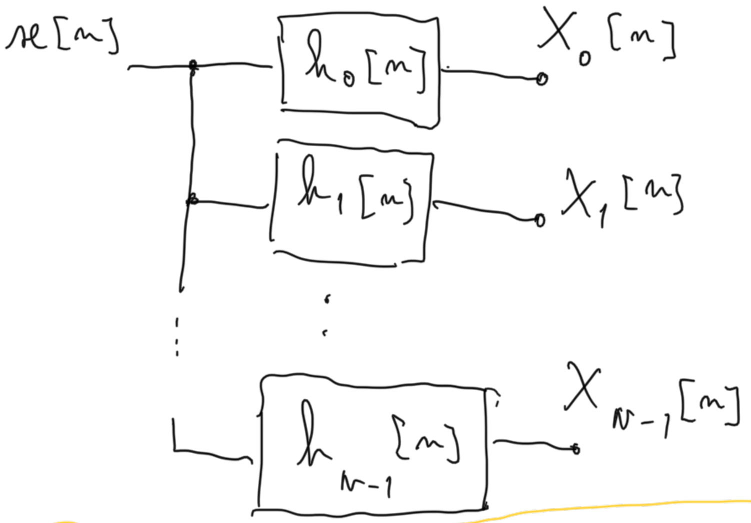
$$n-(N-1) \leq l \leq n$$

$$= \sum_{l=-\infty}^{+\infty} x[l] \, e^{-j\frac{2\pi}{N}(N-1)k} \, w[n-l] \, e^{j\frac{2\pi}{N}(n-l)k}$$

$$= h_k[n-l]$$

$$= \sum_{l=-\infty}^{+\infty} x[l] \, h_k[n-l]$$

$k = 0, 1, \ldots, N-1$




TAKE-HOME MESSAGE:

the DFT is a bank of N parallel filters

$$\omega_k = \frac{2\pi}{N} k$$

$$h_k[n] = w[n] \, e^{-j\frac{2\pi}{N}(N-1)k} \, e^{j\frac{2\pi}{N} n k}$$

question:

how do the filters look like?

$$H_k(e^{j\omega}) \triangleq \sum_{n=-\infty}^{+\infty} h_k[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} w[n]\, e^{-j\frac{2\pi}{N}(N-1)k}\, e^{j\omega_k n}\, e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(N-1)k}\, e^{-j(\omega-\omega_k)n}$$

$$= e^{-j\frac{2\pi}{N}(N-1)k} \sum_{n=0}^{N-1} \left(e^{-j(\omega-\omega_k)}\right)^n$$

$$= \frac{1-e^{-j(\omega-\omega_k)N}}{1-e^{-j(\omega-\omega_k)}}$$

$$= \frac{e^{-j(\omega-\omega_k)\frac{N}{2}}}{e^{-j(\omega-\omega_k)\frac{1}{2}}} \cdot \frac{\cancel{2j}\,\sin(\omega-\omega_k)\frac{N}{2}}{\cancel{2j}\,\sin(\omega-\omega_k)\frac{1}{2}}$$

$$= e^{-j\frac{2\pi}{N}(N-1)k}\, e^{-j(\omega-\omega_k)\frac{N-1}{2}}\, \frac{\sin(\omega-\omega_k)\frac{N}{2}}{\sin(\omega-\omega_k)\frac{1}{2}}$$

$$|H_k(e^{j\omega})| = \left|\sin\left((\omega-\omega_k)\frac{N}{2}\right)\right| \ldots$$

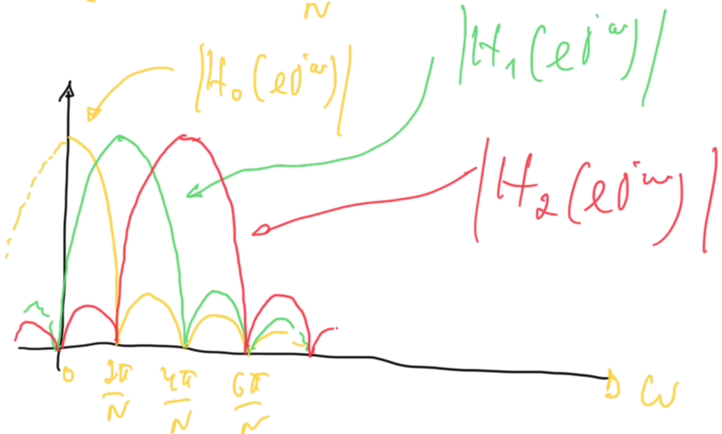
$$|H_K(e^{j\omega})| = \left|\frac{\sin\left((\omega-\omega_K)\frac{N}{2}\right)}{\sin\left((\omega-\omega_K)\frac{1}{2}\right)}\right| \qquad K = 0, 1, \ldots, N-1$$

$$\omega_K\Big|_{K=0} = \frac{2\pi}{N}K = 0$$

$$|H_0(e^{j\omega})| = \left|\frac{\sin\left(\omega\frac{N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right|$$

i.e., the first DFT filter is a periodic sinc function

$$\omega\frac{N}{2} = m\pi \quad \therefore \quad \omega = m\frac{2\pi}{N}$$




question: how about other filters? $K = 1, 2, \ldots, N-1$

for example:

$K=1$, $|H_1(e^{j\omega})| = ?$

$$|H_K(e^{j\omega})| = \left|\frac{\sin\left((\omega-\omega_K)\frac{N}{2}\right)}{\sin\left((\omega-\omega_K)\frac{1}{2}\right)}\right|$$

$$\omega_K = \frac{2\pi}{N}K, \quad K = 0, 1, \ldots, N-1$$

$$K=1 \Rightarrow \omega_1 = \frac{2\pi}{N}$$

$$|H_1(e^{j\omega})| = \left|\sin\left(\omega - \frac{2\pi}{N}\right)\frac{N}{2} \ \ldots\right| = \left|H_0\left(e^{j\left(\omega-\frac{2\pi}{N}\right)}\right)\right|$$

$$\ldots \sin\left(\omega - \frac{2\pi}{N}\right)\frac{1}{2} \ldots$$

$\therefore$ it consists of a frequency-shifted version of $|H_0(e^{j\omega})|$

SECOND TAKE-HOME MESSAGE:

the different filters of the DFT correspond to uniformly frequency-modulated versions of a prototype filter which is the Fourier transform of the window $w[n]$

Idea of signal segmentation in frequency analysis $\equiv$ spectrum estimation


$|X[k]|$ ← "periodogram" or

'magnitude spectrum

when periodograms are looked at as "slices" and are put next to each other (i.e. abutted then we obtain a 3D representation called the spectrogram

(see the Matlab demo on the video recording)