

1.

a)

- i) FALSE: given that the order of a system equals the number of its poles/zeros, then ZP1 and ZP2 have order 4, and ZP3 has order 8.
- ii) TRUE: as all the zeros/poles pertaining to all three systems are located inside the unit circle.
- iii) FALSE: none of the systems has a real-valued impulse response as that would require that all poles and all zeros are either located on the real axis of the z -plane, or are organized as complex-conjugate pairs.
- iv) TRUE: if ZP1 and ZP3 are cascaded, then each pole is cancelled out by a zero which means that the resulting transfer function is simply $H(z) = G$, where G is a constant, which means the resulting system is zero-order.
- v) TRUE: if ZP1 and ZP2 are cascaded, or if ZP3 and ZP2 are cascaded, then 4 zeros are cancelled out by 4 poles that exist at the same locations, but 4 double zeros also appear at specific locations, which means that the resulting system has order 8.

b) The easiest association is ZP1 - FRC. This is because this is the only system having a pole close to $z = e^{j0} = 1$ which causes the magnitude frequency response to peak for $\omega = 0$ rad. The next easy association is ZP3 - FRB. The reason is that the angles of the poles that are closer to the unit circle are $\omega = \pi/4$ and $\omega = 3\pi/4$ rad. which causes the magnitude frequency response to peak at these frequencies. Finally, ZP2 corresponds to FRA

because zeros are closer to the unit circumference between $\omega=0$ and $\omega=\pi$ rad. which causes the magnitude frequency response to become more attenuated in this frequency range.

2. a) $x_e(t) = e^{j420\pi t} + \sin 500\pi t/3 + \cos 800\pi t/3$

$$x[n] = x_e(t) \Big|_{t=\frac{n}{F_s}} = e^{jn\omega_1} + \sin n\omega_2 + \cos n\omega_3$$

where:

$$\omega_1 = \frac{420\pi}{100} = \frac{42\pi}{10} = \frac{21\pi}{5} + K2\pi = \frac{21\pi + K10\pi}{5} \Big|_{K=-2} = \frac{\pi}{5} \text{ rad.}$$

$$\omega_2 = \frac{500\pi}{3 \times 100} = \frac{5\pi}{3} + K2\pi = \frac{5\pi + K6\pi}{3} \Big|_{K=-1} = -\frac{\pi}{3} \text{ rad.}$$

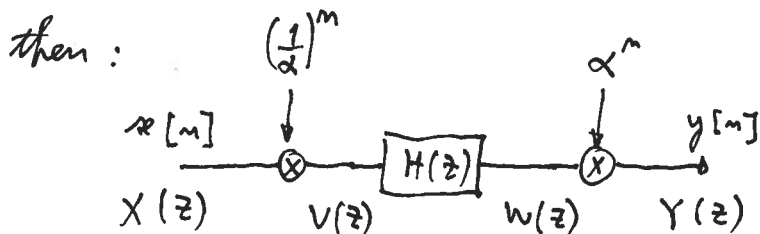
$$\omega_3 = \frac{800\pi}{3 \times 100} = \frac{8\pi}{3} + K2\pi = \frac{8\pi + K6\pi}{3} \Big|_{K=-1} = \frac{2\pi}{3} \text{ rad.}$$

Thus:

$$\begin{aligned} x[n] &= e^{jn\frac{\pi}{5}} + \sin(-n\frac{\pi}{3}) + \cos n\frac{2\pi}{3} \\ &= e^{jn\frac{\pi}{5}} - \sin n\frac{\pi}{3} + \cos n\frac{2\pi}{3} \end{aligned}$$

b) From the Z-Transform properties, we know that

$$\begin{aligned} x[n] &\longleftrightarrow X(z) \quad , R_x \\ \alpha^n x[n] &\longleftrightarrow X\left(\frac{z}{\alpha}\right) \quad , |\alpha| R_x \end{aligned}$$



and: $v(z) = X\left(\frac{z}{1/\alpha}\right) = X(\alpha z)$

$$w(z) = H(z)v(z) = H(z)X(\alpha z)$$

$$Y(z) = W\left(\frac{z}{\alpha}\right) = H\left(\frac{z}{\alpha}\right)X\left(\frac{\alpha z}{\alpha}\right) = H\left(\frac{z}{\alpha}\right)X(z)$$

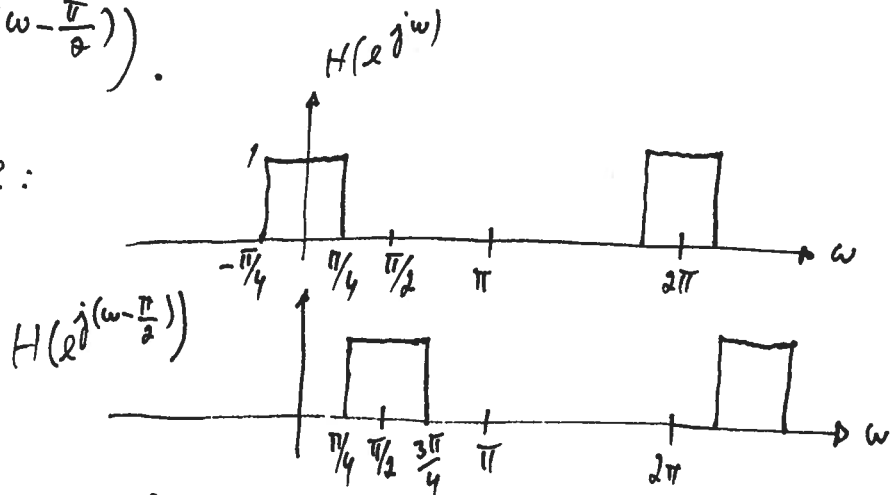
Given that we know that $\alpha = j = e^{j\pi/2}$, then in the Fourier domain we have:

$$Y(e^{j\omega}) = H\left(\frac{e^{j\omega}}{e^{j\pi/2}}\right) X(e^{j\omega}) = H(e^{j(\omega - \pi/2)}) X(e^{j\omega})$$

which means that

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = H(e^{j(\omega - \pi/2)}).$$

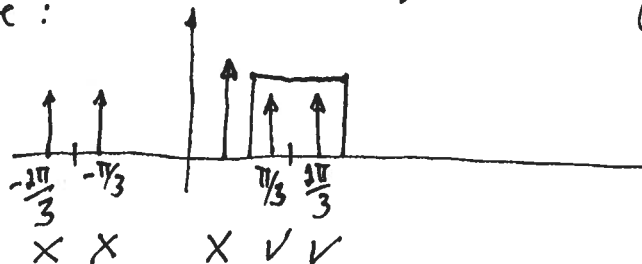
Graphically, we have:



It should be noted that this corresponds to a modulated (i.e. frequency-modulated) version of the initial low-pass filter. The new center frequency is $\pi/2$ rad. Thus, the final (equivalent) filter is an ideal band-pass filter. Due to the fact that it has only one band within one period of the frequency representation (i.e. that band has no mirror image on the negative frequency axis), then the impulse response of the final filter is complex-valued.

c) $x[n]$ can be written as $x[n] = e^{jn\pi/5} - \frac{e^{jn\pi/3} - e^{-jn\pi/3}}{2j} + \frac{e^{j\frac{2\pi}{3}n} - e^{-j\frac{2\pi}{3}n}}{2}$

due to effect of the ideal band-pass filter, only a few frequencies survive:



which leads to $y[n] = -\frac{e^{jn\pi/3}}{2j} + \frac{e^{j2\pi/3n}}{2} = \frac{j}{2} e^{jn\pi/3} + \frac{e^{j2\pi/3n}}{2}$
 $= \frac{1}{2} e^{j(n\pi/3 + \pi/2)} + \frac{1}{2} e^{jn\frac{2\pi}{3}}$

and, considering ideal reconstruction conditions, then

$$y[n] = \frac{1}{2} e^{j\left(\frac{m}{100} \frac{\pi}{3} 100 + \frac{\pi}{2}\right)} + \frac{1}{2} e^{j\frac{m}{100} \frac{2\pi}{3} 100}$$

$$= \frac{1}{2} e^{j\left(100\pi t/3 + \frac{\pi}{2}\right)} + \frac{1}{2} e^{j200\pi t/3} \Big|_{t=\frac{n}{F_s}}$$

$$= y_c(n/F_s) = y_c(t) \Big|_{t=\frac{n}{F_s}}$$

which leads to $y_c(t) = \frac{1}{2} e^{j(100\pi t/3 + \pi/2)} + \frac{1}{2} e^{j200\pi t/3}$.

3. $x[n] \xrightarrow{\text{DFT}} X[K] \quad n, K = 0, 1, \dots, N-1$

a) $Y[K] = Y[N+K] = X[K] \quad K = 0, 1, \dots, N-1$

Both $y[n]$ and $Y[K]$ are $2N$ -periodic, thus

$$y[n] = \frac{1}{2N} \sum_{K=0}^{2N-1} Y[K] W_{2N}^{-Kn}$$

$$= \frac{1}{2N} \sum_{K=0}^{N-1} Y[K] W_{2N}^{-Kn} + \frac{1}{2N} \sum_{K=0}^{N-1} \underbrace{Y[N+K]}_{=X[K]} W_{2N}^{-(K+N)n}$$

$$= \frac{1}{2N} \sum_{K=0}^{N-1} X[K] W_{2N}^{-Kn} + \frac{1}{2N} \sum_{K=0}^{N-1} X[K] W_{2N}^{-Kn} \underbrace{W_{2N}^{-nN}}_{=e^{j\frac{2\pi}{2N} nN} = (-1)^n}$$

Given that by definition

$$x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X[K] W_N^{-Kn}$$

we may evaluate $y[n]$ for n even, and for n odd :

$$y[2m] = \frac{1}{2N} \sum_{K=0}^{N-1} X[K] W_{2N}^{-2Km} + \frac{1}{2N} \sum_{K=0}^{N-1} X[K] W_{2N}^{-2Km} (-1)^{2m}$$

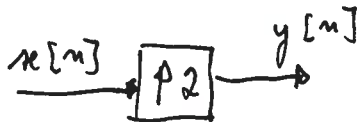
$$= \frac{1}{2N} \sum_{K=0}^{N-1} X[K] W_N^{-Km} + \frac{1}{2N} \sum_{K=0}^{N-1} X[K] W_N^{-Km} = \frac{1}{N} \sum_{K=0}^{N-1} X[K] W_N^{-Km} = x[m]$$

similarly:

$$y[2m+1] = \frac{1}{2N} \sum_{k=0}^{N-1} x[k] W_{2N}^{-k(2m+1)} + \frac{1}{2N} \sum_{k=0}^{N-1} x[k] W_{2N}^{-k(2m+1)} \underbrace{(-1)^{2m+1}}_{=-1, \forall m \in \mathbb{Z}}$$

$$= 0$$

This means that $y[n]$ is an oversampled version of $x[n]$ by a factor of 2, i.e.



b) $f[n] = y[n] + y[(n-1)_{2N}]$, $n = 0, 1, \dots, 2N-1$

Given the DFT properties, we have

$$\begin{aligned} y[n] &\longleftrightarrow Y[K] \\ y[(n-n_0)_{2N}] &\longleftrightarrow e^{-j\frac{2\pi}{2N}K n_0} Y[K] \end{aligned}$$

$$\begin{aligned} f[n] = y[n] + y[(n-1)_{2N}] &\longleftrightarrow Y[K] + e^{-j\frac{\pi}{N}K} Y[K] \\ &= Y[K] \left(1 + e^{-j\frac{\pi}{N}K}\right) = F[K] \end{aligned}$$

thus, $F[K] = \left(1 + e^{-j\frac{\pi}{N}K}\right) Y[K]$, $K = 0, 1, \dots, 2N-1$

c) The Matlab command $Y = [X \ X]$ implements

$$Y[K] = Y[K+N] = X[K], \quad K = 0, 1, \dots, N-1$$

which means that, according to a) $y[2m] = x[m]$ and $y[2m+1] = 0$. Thus, if

$$x = [1 \ 2 \ 3 \ 4 \ 5]$$

then

$$y = \text{ifft}(Y) = [1 \ 0 \ 2 \ 0 \ 3 \ 0 \ 4 \ 0 \ 5 \ 0]$$

d) The third Matlab code implements $F[K] = \left(1 + e^{-j\frac{\pi}{N}K}\right) Y[K]$ which means that $\text{ifft}(F)$ produces

$$f[n] = y[n] + y[(n-1)_{2N}] = [1 \ 1 \ 2 \ 2 \ 3 \ 3 \ 4 \ 4 \ 5 \ 5]$$

e) Using the DFT properties we know that

$$\begin{aligned}
 & \mathcal{R}[n] \xleftrightarrow{\text{DFT}} X[k] \\
 & \mathcal{R}_{\text{ep}}[n] = \frac{\mathcal{R}[n] + \mathcal{R}^*[(n)_N]}{2} \longleftrightarrow \text{Re}\{X[k]\} \\
 & \mathcal{R}_{\text{op}}[n] = \frac{\mathcal{R}[n] - \mathcal{R}^*[(n)_N]}{2} \longleftrightarrow j \text{Im}\{X[k]\} \\
 & \mathcal{R}_{\text{ep}}[n] \circledast \mathcal{R}_{\text{op}}[n] \longleftrightarrow j \text{Im}\{X[k]\} \text{Re}\{X[k]\}
 \end{aligned}$$

Thus:

	$n=0$ ↓				$n=N-1$ ↓
$\mathcal{R}[n]$	1	2	3	4	5
$\mathcal{R}^*[(n)_N]$	1	5	4	3	2
$\mathcal{R}_{\text{ep}}[n]$	1	7/2	7/2	7/2	7/2
$\mathcal{R}_{\text{op}}[n]$	0	-3/2	-1/2	1/2	3/2

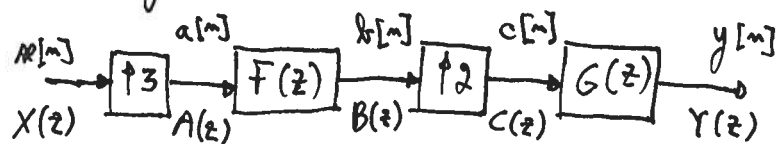
Now, performing the circular convolution:

$$\mathcal{R}_{\text{ep}}[n] \circledast \mathcal{R}_{\text{op}}[n] = \mathcal{R}_{\text{op}}[n] \circledast \mathcal{R}_{\text{ep}}[n] = \sum_{k=0}^{N-1} \mathcal{R}_{\text{op}}[k] \mathcal{R}_{\text{ep}}[(n-k)_N]$$

	$n=0$ ↓				$n=N-1$ ↓	
$\mathcal{R}_{\text{op}}[k]$	0	-3/2	-1/2	1/2	3/2	
$\mathcal{R}_{\text{ep}}[(-k)_N]$	1	7/2	7/2	7/2	7/2	→ 0
$\mathcal{R}_{\text{ep}}[(1-k)_N]$	7/2	1	7/2	7/2	7/2	→ 3/2(7/2-1) = 15/4
$\mathcal{R}_{\text{ep}}[(2-k)_N]$	7/2	7/2	1	7/2	7/2	→ 1/2(7/2-1) = 5/4
$\mathcal{R}_{\text{ep}}[(3-k)_N]$	7/2	7/2	7/2	1	7/2	→ 1/2(1-7/2) = -5/4
$\mathcal{R}_{\text{ep}}[(4-k)_N]$	7/2	7/2	7/2	7/2	1	→ 3/2(1-7/2) = -15/4

which means that the $\text{ifft}(j * \text{imag}(x) * \text{real}(x))$
 is $\left[0 \quad \frac{15}{4} \quad \frac{5}{4} \quad -\frac{5}{4} \quad -\frac{15}{4}\right]$

4. Taking the system on the left:



and using a z -domain analysis involving oversampling and interpolation, we obtain:

$$A(z) = X(z^3)$$

$$B(z) = F(z)A(z) = F(z)X(z^3)$$

$$C(z) = B(z^2) = F(z^2)X((z^2)^3) = F(z^2)X(z^6)$$

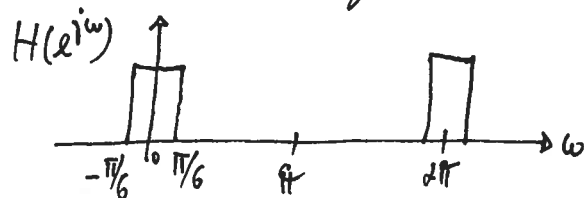
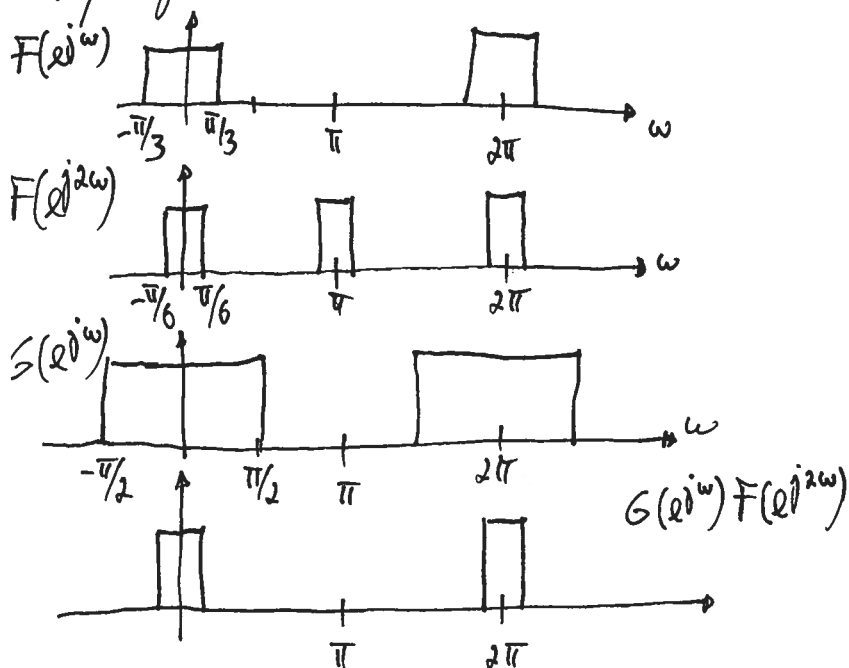
$$Y(z) = G(z)C(z) = G(z)F(z^2)X(z^6)$$

On the other hand, an analysis of the system on the right leads to:

$$Y(z) = H(z)X(z^6)$$

Combining the two results we obtain $H(z) = G(z)F(z^2)$ which, in the Fourier domain reduces to $H(e^{j\omega}) = G(e^{j\omega})F(e^{j2\omega})$

In order to compare the two systems, we consider the filter specifications.



This means that the system on the right-hand side uses a filter whose specifications are critical, especially because the bandwidth of the pass-band is small and the transition between pass and stop band must be short.

This means that, in general, when the oversampling factor is high, the design of the interpolation filter is difficult and, most likely, the filter will be high-order.

This constitutes the advantage of the system on the left: it represents a "divide-and-conquer" strategy in the sense that the final filter $G(e^{j\omega})F(e^{j2\omega})$ is obtained at the cost of two filters, $G(e^{j\omega})$ and $F(e^{j\omega})$, whose designs are simpler (than that of $H(e^{j\omega})$) and whose specifications can be more relaxed. A good final filter benefits from the frequency compression that affects $F(e^{j2\omega})$. One inconvenient is that the implementation complexity of the system on the left may be higher (than that of the system on the right).

Another possible disadvantage is that the oversampling that follows $F(z)$ has the effect to increase the length of the filter impulse response which amplifies its group delay (assuming we are dealing with linear-phase filters).

[5.] a) By inspection, we obtain directly from the realization structure that

$$y[n] = x[n] + 0.8x[n-1] + 0.64x[n-2] + 0.25y[n-2]$$

which leads to

$$Y(z) = X(z) + 0.8z^{-1}X(z) + 0.64z^{-2}X(z) + 0.25z^{-2}Y(z)$$

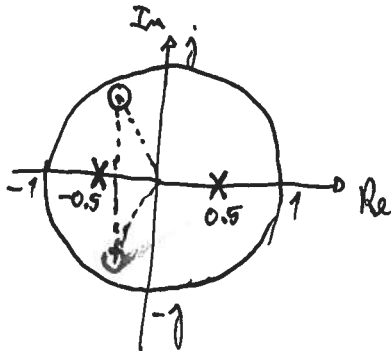
and
$$\frac{Y(z)}{X(z)} = \frac{1 + 0.8z^{-1} + 0.64z^{-2}}{1 - 0.25z^{-2}}, \text{ causal}$$

b) Zeros:
$$z = \frac{-0.8 \pm \sqrt{0.8^2 - 4 \times 0.64}}{2} = \frac{-0.8 \pm 0.8\sqrt{-3}}{2}$$
$$= 0.8 \left(-\frac{1}{2} \pm j \frac{\sqrt{3}}{2} \right) = 0.8 e^{\pm j 2\pi/3}$$

Poles: $(1 - 0.25z^{-2}) = (1 - 0.5z^{-1})(1 + 0.5z^{-1})$

$\therefore z = \pm \frac{1}{2}$

Zero-pole diagram:



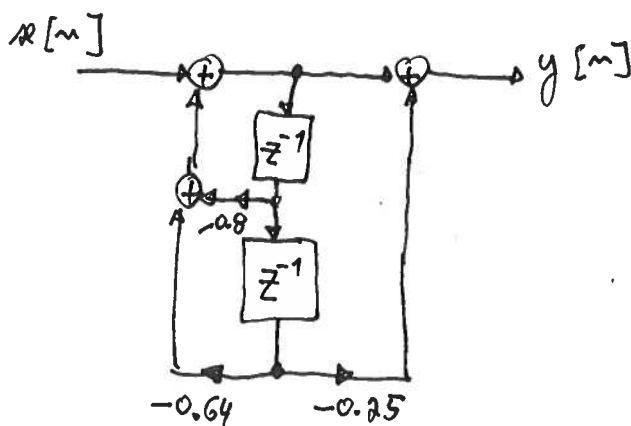
c) The inverse system is given by the transfer function

$$H_i(z) = \frac{1}{H(z)} = \frac{1 - 0.25z^{-2}}{1 + 0.8z^{-1} + 0.64z^{-2}}, \quad |z| > 0.8$$

the corresponding difference equation is

$$y[n] = x[n] - 0.25x[n-2] - 0.8y[n-1] - 0.64y[n-2]$$

and a canonic direct type-2 realization structure is:



6. This is an open question. Students should address that:

- two methods have been studied: the impulse invariance method (II) and the bilinear transformation (BT)
- the II is very simple since the impulse response of the discrete-time filter is obtained by just sampling the impulse response of the analog filter: $h[n] = T h_c(nT)$

- the BT allows to obtain the transfer function of the discrete-time system (or filter) by using the transfer function of the analog filter (in the Laplace domain) and by using a variable transformation (bilinear):

$$H(z) = H_c(s) \Big|_{s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}$$

- the II method suffers from the aliasing phenomenon which makes that the frequency response of the discrete-time filter is always a degraded version of the initial analog filter frequency response; in particular high-pass filters cannot be designed using the II method
- in contrast, the BT method does not suffer from aliasing but establishes a non-linear relationship between the analog frequency (Ω) and the "digital" frequency (ω); this is not a problem in the case of piecewise multiband filter designs but may introduce frequency warping distortions in the case of special filters such as differentiators.

Students aiming at complete answers should also address the mapping of the s-plane into the z-plane, in the case of each method, as well as its consequences.