

NOTE: These solutions are didactic on purpose, as such, they are more detailed than what is required for a Student to write on the exam, however, Students must always provide clear and complete answers, including at the level of Portuguese or English expression.

1.

a) $C \leftrightarrow FR2$

This is the easiest association to identify because the zero-pole diagram C consists of 4 pairs of zero-pole where each zero is located at the reciprocal-conjugate position of the pole it pairs with, which means that each zero-pole pair implements a first-order all-pass, therefore, the whole system is all-pass which means that the frequency response magnitude is constant, and this corresponds to $FR2$.

Systems A and B consist of an all-pole and all-zero system, respectively, both of order 4, and their pole and zero distributions implement an IIR Comb filter, and an FIR Comb filter, respectively as it was discussed in the FPS LAB #5. During this LAB class, it was concluded that IIR Comb filters are good at emphasizing peaks in the magnitude frequency response, and that FIR comb filters are good at emphasizing valleys. This is because IIR Comb filters take advantage of the proximity between the poles and the unit circumference, whereas FIR Comb filters take advantage of the proximity between the zeros and the unit circumference. This leads to the associations:

$A \leftrightarrow FR3$ and $B \leftrightarrow FR1$

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b) System A: is IIR because zeros are located in the special positions $z=0$ or $z=\infty$ while poles are not, which means that this system is all-pole

System B: is FIR because poles are located in the special (i.e. agnostic) positions $z=0$ or $z=\infty$ while zeros are not, which means that this system is all-zero

System C: is IIR because the poles are not all located in the special z positions and are not cancelled out by zeros

A system is minimum-phase if its zeros and poles are all located inside the unit circle, which is true for systems A and B. Therefore, only system C is not minimum-phase.

c) In order to answer this question, it is necessary to see if there is a way to relocate the poles and zeros of systems A and B such that they match the poles and zeros of system C. It is clear from the zero-pole diagrams that the poles of system A are the same as the poles of system C. On the other hand, if the zeros of system B are taken to their reciprocal-conjugate locations (which can be achieved by taking $H_B^*(1/z^*)$ instead of just $H_B(z)$), and if that configuration is left or right-rotated by $\pi/4$, then the resulting zeros match those of system C. Since rotation is achieved by changing z to $z/e^{\pm j\pi/4}$,

then, system B must be transformed as:

$$H_B(z) \rightarrow H_B^* \left(\frac{e^{\mp j\pi/4}}{z^*} \right)$$

In addition, because this transformation makes that all the poles of the original system B are relocated to infinity, in order to bring them back to $z=0$ (such that they cancel the zeros of system A), then it is necessary to multiply the overall transfer function by z^{-4} . Thus, we have

$$H_C(z) = z^{-4} H_A(z) H_B^* \left(\frac{e^{\mp j\pi/4}}{z^*} \right)$$

NOTE: $H_C(z) = z^{-4} H_A(z) H_B \left(\frac{e^{\pm j\pi/4}}{z} \right)$ also works, why?

NOTE 2: $H_C(z)$ could also be expressed as a function of $H_A(z)$, how?

2.

a) $F_S = 120 \text{ Hz}$

$$y[n] = x[n] - x[n-3]$$

$$x_c(t) = 1 + \sin(80\pi t) + \sin(540\pi t)$$

Considering ideal sampling:

$$x[n] = x_c(t) \Big|_{t=nT=\frac{n}{F_S}} = 1 + \sin\left(80\pi \frac{n}{120}\right) + \sin\left(540\pi \frac{n}{120}\right)$$

\uparrow
 ω_0

\uparrow
 ω_1

\uparrow
 ω_2

$$= 1 + \sin(n\omega_1) + \sin(n\omega_2)$$

and:

$$\omega_0 = 0 \text{ rad}$$

$$\omega_1 = \frac{8\pi}{12} = \frac{2\pi}{3} \text{ rad.} < \pi$$

$$\omega_2 = \frac{54\pi}{12} = \frac{2 \times 3 \times 9}{2 \times 2 \times 3} \pi = \frac{9\pi}{2} > \pi \text{ which means that}$$

due to the aliasing phenomenon:

$$\omega_2 = \frac{9\pi}{2} + k2\pi = \frac{9\pi + k8\pi}{2} \Big|_{k=-1} = \frac{\pi}{2} \text{ rad}$$

this leads to:

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$$x[n] = 1 + \sin\left(n\frac{2\pi}{3}\right) + \sin\left(n\frac{\pi}{2}\right)$$

b) Given that $y[n] = x[n] - x[n-3]$, taking the Fourier Transform we obtain

$$Y(e^{j\omega}) = X(e^{j\omega}) - e^{-j3\omega} X(e^{j\omega}) \quad \therefore$$

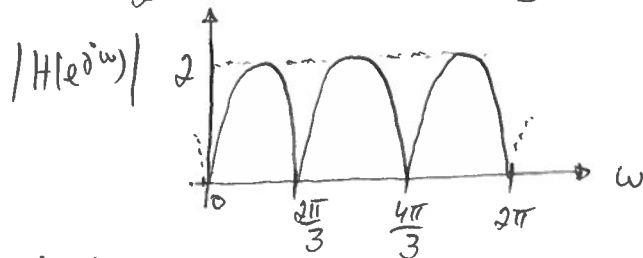
$$\begin{aligned} \frac{Y(e^{j\omega})}{X(e^{j\omega})} = H(e^{j\omega}) &= 1 - e^{-j3\omega} = e^{-j\frac{3}{2}\omega} (e^{j\frac{3}{2}\omega} - e^{-j\frac{3}{2}\omega}) \\ &= e^{-j\frac{3}{2}\omega} 2j \sin\left(\frac{3}{2}\omega\right) = e^{-j\frac{3}{2}\omega} e^{j\frac{\pi}{2}} 2 \sin\left(\frac{3}{2}\omega\right) \\ &= e^{j(\frac{\pi}{2} - \frac{3\omega}{2})} 2 \sin\left(\frac{3}{2}\omega\right) \end{aligned}$$

This means that

$$|H(e^{j\omega})| = 2 \left| \sin\left(\frac{3}{2}\omega\right) \right|$$

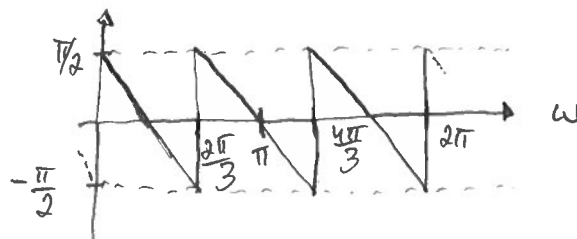
$\angle H(e^{j\omega}) = \frac{\pi}{2} - \frac{3\omega}{2} + \text{jumps of } \pi \text{ when } \sin\left(\frac{3\omega}{2}\right) \text{ crosses zero and switches polarity}$

Now, $\sin\frac{3\omega}{2} = 0 \Rightarrow \frac{3\omega}{2} = k\pi \therefore \omega = k\frac{2\pi}{3}$, which leads to:



polarity of $\sin\frac{3\omega}{2} \rightarrow \oplus \quad \ominus \quad \oplus$

and the phase response becomes:



c) Given that the frequencies in $x[n]$ are ω_0 , ω_1 and ω_2 , then we have to find first:

$$H(e^{j\omega})|_{\omega=\omega_0} = 0$$

$$H(e^{j\omega})|_{\omega=\omega_1} = 0$$

$$H(e^{j\omega})|_{\omega=\omega_2} = e^{j(\frac{\pi}{2} - \frac{3\omega}{2})} 2 \sin \frac{3\omega}{2} \Big|_{\omega=\frac{\pi}{2}} = e^{j(\frac{\pi}{2} - \frac{3\pi}{4})} 2 \sin \frac{3\pi}{4} = e^{-j\frac{\pi}{4}} \sqrt{2}$$

which leads to:

$$y[n] = |H(e^{j\frac{\pi}{2}})| \sin(n\frac{\pi}{2} + \angle H(e^{j\frac{\pi}{2}})) = \sqrt{2} \sin(n\frac{\pi}{2} - \frac{\pi}{4})$$

Presuming ideal reconstruction conditions

$y[n]$ can be looked at as

$$y[n] = y_c(t) \Big|_{t=\frac{n}{F_s}} = \sqrt{2} \sin\left(\frac{n}{F_s} F_s \frac{\pi}{2} - \frac{\pi}{4}\right) = \sqrt{2} \sin\left(60\pi t - \frac{\pi}{4}\right) \Big|_{t=\frac{n}{F_s}}$$

and, therefore, we may say that

$$y_c(t) = \sqrt{2} \sin\left(60\pi t - \frac{\pi}{4}\right)$$

NOTE: it is WRONG to write $y_c(t) = y[n] \Big|_{n=tF_s}$
(see solutions to EXAM PDSI of June 4th 2018, page 4/10)

a) First, this code reads a sample from the left channel (rx_sample_L) and then adds to it a scaled version of the oldest sample stored in buffer $w[]$:

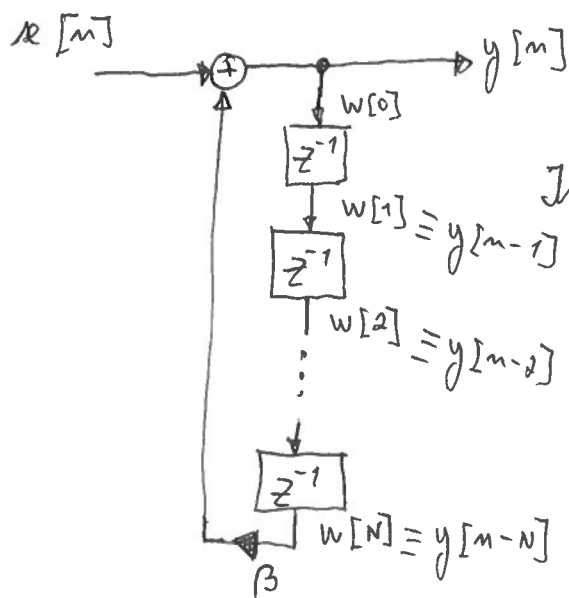
$$y_n = rx_sample_L + \beta w[N]$$

the result is then taken as the output sample of the left channel. Finally, the first sample of the $w[]$ buffer is updated with the value of y_n and all the samples in this buffer (which is a delay line) are right-shifted by one sample which makes the buffer ready to be used in the next interrupt (i.e. when the next input sample arrives).

The code implements the following difference equation:

$$y[n] = x[n] + \beta y[n-N]$$

b) Realizing that the $w[]$ buffer stores all past N values of $y[n]$, the implementation structure emerges directly from the difference equation:



In the z -domain:

$$Y(z) = X(z) + \beta z^{-N} Y(z) \therefore$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{1 - \beta z^{-N}} \quad , \quad |z| > \sqrt[N]{\beta}$$

The poles are found using

$$1 - \beta z^{-N} = 0 \Leftrightarrow z^N = \beta$$

$$\Leftrightarrow z^N = \beta e^{jK2\pi}, \quad K \in \mathbb{Z}$$

and, admitting that β is a positive constant, then $z = \sqrt[N]{\beta} e^{jK\frac{2\pi}{N}}$, $K=0, 1, \dots, N-1$, which means that the RoC is $|z| > \sqrt[N]{\beta}$

4.

a) It is known that:

$$x[n] \xleftrightarrow{\text{DFT}} X[k]$$

$$x^*[n] \xleftrightarrow{\text{DFT}} X^*[-k]_N$$

$$x[-n] \xleftrightarrow{\text{DFT}} X[-k]_N$$

$$x^*[-n] \xleftrightarrow{\text{DFT}} X^*[k]$$

and recognizing that the code implements:

$$Y[k] = X^*[-k]_N$$

and also:

$$X_e[k] = \frac{X[k] + Y[k]}{2} = \frac{X[k] + X^*[-k]_N}{2} \xleftrightarrow{\text{DFT}} \frac{x[n] + x^*[n]}{2} = \text{Re}\{x[n]\}$$

and:

$$X_o[k] = \frac{X[k] - Y[k]}{2} = \frac{X[k] - X^*[-k]_N}{2} \xleftrightarrow{\text{DFT}} \frac{x[n] - x^*[n]}{2} = j \text{Im}\{x[n]\}$$

now, finally:

$$\begin{aligned} \text{DFT}^{-1}\{jX_o[k]\} &= j \text{DFT}^{-1}\{X_o[k]\} = j \times j \text{Im}\{x[n]\} \\ &= -1 \times \text{Im}\{x[n]\} \equiv [0 \ 0 \ 0 \ 1 \ 2 \ 3] \end{aligned}$$

$$\text{because } x[n] \equiv [3 \ 2 \ 1 \ -j \ -2j \ -3j]$$

$$\text{and } \text{Re}\{x[n]\} \equiv [3 \ 2 \ 1 \ 0 \ 0 \ 0]$$

$$\text{and } \text{Im}\{x[n]\} \equiv [0 \ 0 \ 0 \ -1 \ -2 \ -3]$$

b) It is known that product in the DFT domain corresponds to circular convolution in the discrete (and periodic) n domain, i.e. if

$$a[n] \xleftrightarrow{\text{DFT}} A[k]$$

$$b[n] \xleftrightarrow{\text{DFT}} B[k]$$

$$z[n] = a[n] \circledast b[n] \xleftrightarrow{\text{DFT}} A[k] B[k]$$

which means that in our case :

$$a[n] = \Re\{x[n]\} \equiv [3 \ 2 \ 1 \ 0 \ 0 \ 0] \xleftrightarrow{\text{DFT}} X_e[k]$$

$$b[n] = -\Im\{x[n]\} \equiv [0 \ 0 \ 0 \ 1 \ 2 \ 3] \xleftrightarrow{\text{DFT}} jX_o[k]$$

and, finding $a[n] \circledast b[n]$:

$a[n]$	\dots	0	3	2	1	0	0	0	3	\dots
			$\downarrow m=0$						$\downarrow m=N-1$	
$b[n]$	\dots	3	0	0	0	1	2	3	0	\dots
			$\uparrow m=0$					$\uparrow m=N-1$		
$b[(1-m)_N]$		0	3	2	1	0	0		$\therefore z[0] = 8$	
$b[(1-m)_N]$		0	0	3	2	1	0		$\therefore z[1] = 3$	
$b[(2-m)_N]$		0	0	0	3	2	1		$\therefore z[2] = 0$	
$b[(3-m)_N]$		1	0	0	0	3	2		$\therefore z[3] = 3$	
$b[(4-m)_N]$		2	1	0	0	0	3		$\therefore z[4] = 8$	
$b[(5-m)_N]$		3	2	1	0	0	0		$\therefore z[5] = 14$	

which means that $a[n] \circledast b[n] = \sum_{k=0}^{N-1} a[k] b[(n-k)_N]$

$$\equiv [8 \ 3 \ 0 \ 3 \ 8 \ 14]$$

5. a) $H(e^{j\omega}) = \begin{cases} e^{j\theta}, & -\pi \leq \omega < 0 \\ e^{-j\theta}, & 0 \leq \omega < \pi \end{cases}$

By definition,
$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^0 e^{j\theta} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_0^{\pi} e^{-j\theta} e^{j\omega n} d\omega$$

$$\begin{aligned}
&= \frac{e^{j\theta}}{2\pi} \left[\frac{e^{j\omega n}}{j^n} \right]_{-\pi}^0 + \frac{e^{-j\theta}}{2\pi} \left[\frac{e^{j\omega n}}{j^n} \right]_0^{\pi} \\
&= \frac{e^{j\theta}}{2\pi} \frac{1 - e^{-j\omega n\pi}}{j^n} + \frac{e^{-j\theta}}{2\pi} \frac{e^{j\omega n\pi} - 1}{j^n} \\
&= \frac{e^{j\theta}}{2\pi} e^{j\omega n\frac{\pi}{2}} \frac{e^{j\omega n\frac{\pi}{2}} - e^{-j\omega n\frac{\pi}{2}}}{j^n} + \frac{e^{-j\theta}}{2\pi} e^{j\omega n\frac{\pi}{2}} \frac{e^{j\omega n\frac{\pi}{2}} - e^{-j\omega n\frac{\pi}{2}}}{j^n} \\
&= \frac{e^{j(\theta - n\frac{\pi}{2})}}{2\pi} \frac{2j \sin n\frac{\pi}{2}}{j^n} + \frac{e^{-j(\theta - n\frac{\pi}{2})}}{2\pi} \frac{2j \sin n\frac{\pi}{2}}{j^n} \\
&= \frac{\sin n\frac{\pi}{2}}{n\pi} \left(e^{j(\theta - n\frac{\pi}{2})} + e^{-j(\theta - n\frac{\pi}{2})} \right) \\
&= 2 \frac{\sin n\frac{\pi}{2}}{n\pi} \cos(\theta - n\frac{\pi}{2}) = \text{sinc} \frac{n\pi}{2} \cos(\theta - n\frac{\pi}{2})
\end{aligned}$$

NOTE: here, we assume that $\text{sinc} \alpha = \frac{\sin \alpha}{\alpha}$

Given that $\lim_{\alpha \rightarrow 0} \text{sinc} \alpha = 1$, then

$h[0] = \cos \theta$, NOTE: we could also use l'Hôpital rule

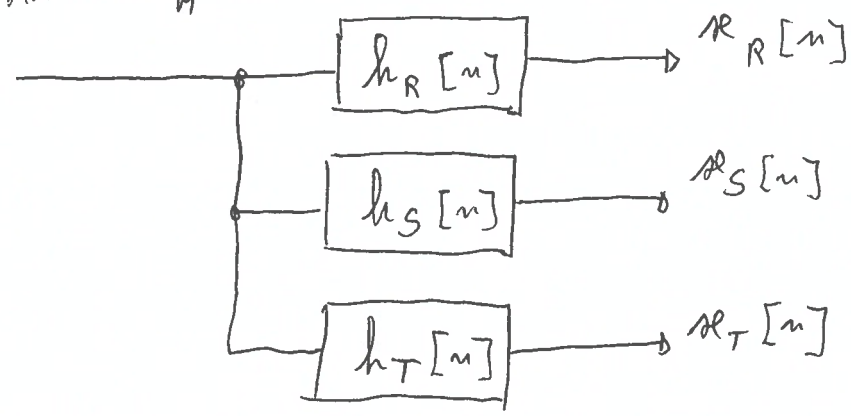
thus: $h[n] = \begin{cases} \cos \theta, & n = 0 \\ \text{sinc} n\frac{\pi}{2} \cos(\theta - n\frac{\pi}{2}), & n \neq 0 \end{cases}$

b) A three-phase system is characterized by the fact that the relative phases of $x_R[n]$, $x_S[n]$ and $x_T[n]$ are evenly distributed in the range $[0, 2\pi]$, this means that if $x_R[n]$ is the reference signal, for example, then the phase of $x_S[n]$ relative

to $x_R[n]$ should be $2\pi/3$, whereas the phase of $x_T[n]$ relative to $x_R[n]$ should be $-2\pi/3$ rad.

Hence, a three-phase system could be obtained by implementing the following processing:

$$x_M[n] = \sin n\omega_m$$



where $h_S[n]$ and $h_T[n]$ are FIR versions of the arbitrary phase shifter obtained in a)

with
$$h_S[n] = h[n] w[n] \Big|_{\theta = \frac{2\pi}{3}}, \quad m = -M, -M+1, \dots, -1, 0, 1, \dots, M-1, M$$

and
$$h_T[n] = h[n] w[n] \Big|_{\theta = -\frac{2\pi}{3}}, \quad m = -M, \dots, -1, 0, 1, \dots, M$$

where $w[n]$ represents a symmetric window having length $2M+1$, and $h[n]$ is the solution obtained in a).

In order for these filters to be causal, their impulse responses must be delayed by M samples which represents the group delay of the filters. As such, in order for the relative phases to be preserved, $x_R[m] = x_M[m-M]$ which means that $h_R[n] = \delta[n-M]$. This solution works in Europe (50Hz) and in North America because the relative phases are preserved independently of the frequency.

6.

a) If the illustrated spectrogram represents the Nyquist frequency range, then the Nyquist frequency is 8 KHz, and the sampling frequency is 16 KHz. It is clear from the spectrogram that two signals exist:

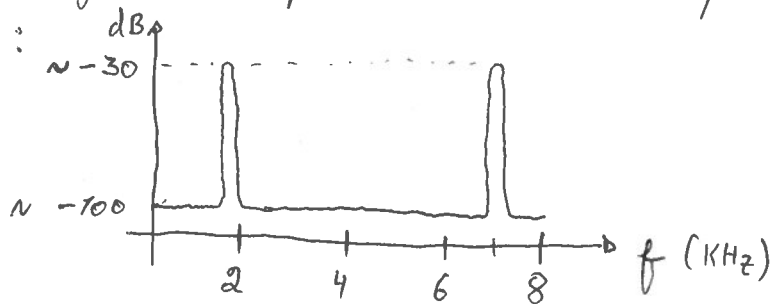
→ a narrow-band signal, most likely a sinusoid, whose center frequency is around 4.5 KHz for $t=0.0s$, and which increases linearly up to the Nyquist frequency; given that this signal is real-valued, given that an anti-aliasing filter does not exist, and given that the signal evolves monotonously (i.e. its frequency evolves smoothly either by increasing, or decreasing, without discontinuous variations), then, once the frequency crosses the Nyquist limit, then the frequency folds back to the Nyquist range (i.e. it suffers aliasing), and what the spectrogram shows is the mirror of the original frequency with respect to the Nyquist frequency, that is why it decreases in the spectrogram, and the process repeats once the aliased frequency reaches zero Hz; thus, we can say that one signal appears to consist of a sinusoid whose frequency increases linearly with time at a rate of $(8+3.5)/0.4 = 28.75$ KHz per second.

→ another narrow-band signal, most likely a sinusoid, whose center frequency is around 3 KHz for $t=0.0s$, and which evolves in

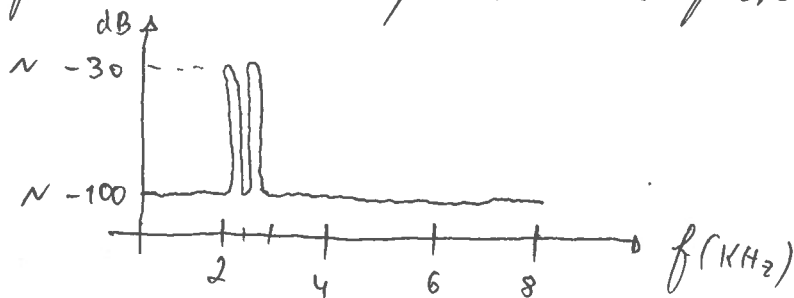
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time in a sinusoidal fashion, in other words, it consists of a frequency-modulated sinusoid according to sinusoidal variation pattern between around 1 KHz and around 3 KHz and whose variation period is 400 ms.

b) A periodogram consists of a vertical line in the spectrogram, for $t = 0.1$ s a plausible representation is:



and for $t = 0.5$ s a plausible representation is:



(a single peak at $f \approx 2.5$ KHz is also acceptable)

c) If an anti-aliasing filter exists, then, if an input frequency exceeds the Nyquist limit, it becomes blocked and does not show up in the spectrogram; according to the explanation in a), the new spectrogram becomes:

