FunSP, EXAM OF JANUARY 19, 2024
— SOLUTION TOPICS —

1.

a) B − 2

This is the easiest association to establish because the two zeros that exist at $z = e^{j0} = 1$ and $z = e^{j\frac{2\pi}{3}}$ cause the frequency response magnitude to become zero for $\omega = 0$ rad. and $\omega = \frac{2\pi}{3}$ rad.

A − 3

This is the second easiest association to identify because not only the two zeros at $z = e^{\pm j\frac{\pi}{3}}$ cause nulls in the frequency response magnitude for $\omega = \pm\frac{\pi}{3}$ rad, but also because the close proximity between those zeros and the two poles oriented for the same angles ($\pm\pi/3$) cause those nulls to be very sharp, a profile that is known in the literature as "notch filtering".

C − 1

The four zeros oriented for $\omega = \pm\frac{\pi}{3}$ rad and close to the unit circumference (two of them over) cause the frequency response magnitude to be very small (or even zero) for frequencies in the vicinity of $\pm\frac{\pi}{3}$ rad

b) Given that all systems are causal and all of the poles are located inside the unit circle, that means that the unit circumference belongs to the region of convergence and, therefore, all systems are stable.
In order for a system to have a real-valued impulse response, all of the zeros and poles of its transfer function must be located on the real axis of the Z plane, or exist as complex-conjugate pairs in that plane. That is the case of systems A and C (and not of system B).

c) In order for a system to have a linear-phase response (meaning that its frequency response has a phase that is a linear function of the frequency), it must be FIR, which means that all poles must be located either at $z=0$ or $z=\infty$, and all zeros must be organized as reciprocal-conjugate pairs, or particular cases of this arrangement, e.g., when they are located over the unit circumference. These conditions are met only in the case of the AC cascade because two zero-pole cancellations occur (making that all poles exist at $z=0$) and 4 zeros exist over the unit circumference (two at $z=e^{j\pi/3}$ and two at $z=e^{-j\pi/3}$). Although it was not requested, we can also say that the order of the resulting system is four, and its impulse response is real-valued because all zeros exist as complex-conjugate pairs.

d) By looking at the A and B zero-pole diagrams, it becomes apparent (i.e., evident) that system A may be obtained by rotating the zero-pole diagram of system B in a clockwise manner by $\pi/3$ rad. This is equivalent to advancing the corresponding frequency response by $\pi/3$ rad. Using the Fourier Transform properties:

$$h_B[n] \xrightarrow{\mathcal{F}} H_B(e^{j\omega})$$

$$h_A[n] = e^{-jn\frac{\pi}{3}} h_B[n] \longleftrightarrow H_A(e^{j\omega}) = H_B\left(e^{j(\omega+\frac{\pi}{3})}\right)$$

So, the statement is true.

$\boxed{2.}$

a) The simplest way to write down the transfer function of system **C** is:

$$H_C(z) = \frac{(z-e^{j\pi/3})(z-e^{-j\pi/3})(z-0.8e^{j\pi/3})(z-0.8e^{-j\pi/3})}{z^4} =$$

$$= \left(1 - e^{j\pi/3} z^{-1}\right)\left(1 - e^{-j\pi/3} z^{-1}\right)\left(1 - 0.8 e^{j\pi/3} z^{-1}\right)\left(1 - 0.8 e^{-j\pi/3} z^{-1}\right)$$

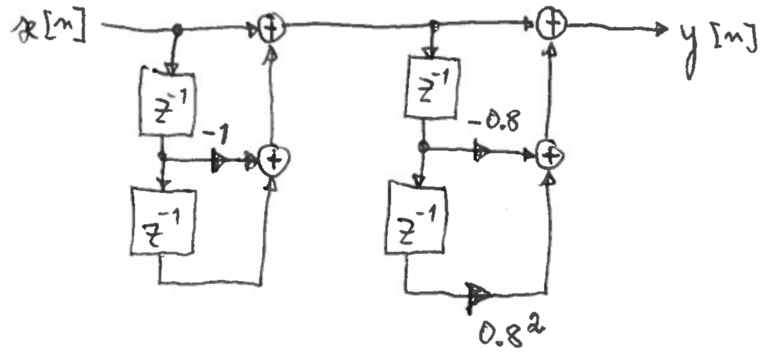
and, by combining the pairs of complex-conjugate zeros:

$$H_c(z) = \left(1 - 2\cos\frac{\pi}{3} z^{-1} + z^{-2}\right)\left(1 - 2 \times 0.8 \times \cos\frac{\pi}{3} z^{-1} + 0.8^2 z^{-2}\right)$$

$$= \left(1 - z^{-1} + z^{-2}\right)\left(1 - 0.8 z^{-1} + 0.8^2 z^{-2}\right) = H_1(z)\, H_2(z)$$

with $H_1(z) = 1 - z^{-1} + z^{-2}$ and $H_2(z) = 1 - 0.8 z^{-1} + 0.8^2 z^{-2}$

a possible canonic realization structure is:



b) If $H(z) = 1 - z^{-1} + z^{-2}$ then, $H(e^{j\omega}) = H(z)\Big|_{z = e^{j\omega}} = 1 - e^{-j\omega} + e^{-j2\omega}$

$$= e^{-j\omega}\left(e^{j\omega} - 1 + e^{-j\omega}\right) = e^{-j\omega}(2\cos\omega - 1)$$

which means that:

$|H(e^{j\omega})| = |2\cos\omega - 1|$ and, in particular:

$$\left|H(e^{j\omega})\right|\Big|_{\omega = 0} = |H(1)| = |2 - 1| = 1$$

$$\left|H(e^{j\omega})\right|\Big|_{\omega = \pi} = |-2 - 1| = 3$$

c) $F_S = 100\,Hz$

$x_c(t) = 1 + \sin(350\pi t)$

$$x[n] = x_c(t)\Big|_{t = nT = \frac{n}{F_S}} = 1 + \sin\frac{350\pi n}{100} = \underset{\uparrow\ \omega_0}{1} + \sin n \underset{\uparrow\ \omega_1}{\frac{7\pi}{2}}$$

This means that two frequencies exist:

$\omega_0 = 0$ rad

$\omega_1 = \frac{7\pi}{2}$ rad. $> \pi$

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As $|\omega_1| > \pi$ aliasing will occur and, therefore,

$$\omega_1 = \frac{7\pi}{2} + k2\pi = \left.\frac{7\pi + k4\pi}{2}\right|_{k=-2} = -\frac{\pi}{2} \text{ rad.}$$

Finally: $x[n] = 1 + \sin(-n\frac{\pi}{2}) = 1 - \sin(n\frac{\pi}{2})$

d) Presuming ideal reconstruction conditions:

$$y[n] = y_c(t)\Big|_{t=\frac{n}{F_s}} = H(e^{j0}) - \left|H(e^{j\frac{\pi}{2}})\right| \sin\left(n\frac{\pi}{2} + \angle H(e^{j\frac{\pi}{2}})\right)$$

as $H(e^{j\omega})\Big|_{\omega=0} = 1$ and $\left|H(e^{j\omega})\right|_{\omega=\frac{\pi}{2}} = \left|2\cos\frac{\pi}{2} - 1\right| = 1$

we have: $y[n] = 1 - \sin\left(\frac{n}{F_s}\frac{F_s\pi}{2} + \angle H(e^{j\pi/2})\right)$

$$= 1 - \sin\left(50\pi t + \angle H(e^{j\pi/2})\right)\Big|_{t=\frac{n}{F_s}}$$

which means that the frequencies that exist in $y_c(t)$ are 0 Hz and 25 Hz and their magnitudes are 1.

$\boxed{3}$

a) Considering that $x[n] \xleftrightarrow{DFT} X[k]$

$$x_{ep}[n] = \frac{x[n] + x^*[(-n)_N]}{2} \xleftrightarrow{DFT} \mathrm{Re}\{X[k]\}$$

$$x_{op}[n] = \frac{x[n] - x^*[(-n)_N]}{2} \longleftrightarrow j\,\mathrm{Im}\{X[k]\}$$

then, $A[k] = \mathrm{Re}\{X[k]\} \xleftrightarrow{DFT} a[n] = \mathrm{ifft}\{A[k]\} = x_{ep}[n]$

and $B[k] = j\,\mathrm{Im}\{X[k]\} \longleftrightarrow b[n] = \mathrm{ifft}\{B[k]\} = x_{op}[n]$

as $x[n]: \underset{\uparrow\, n=0}{j} \;\; 2 \;\; 3j \;\; \underset{\uparrow\, n=N-1}{4}$ then: $x_{ep}[n] \equiv [0 \;\; 3 \;\; 0 \;\; 3] \equiv a[n]$

$x^*[(-n)_N]: \underset{\uparrow\, n=0}{-j} \;\; 4 \;\; -3j \;\; \underset{\uparrow\, n=N-1}{2}$ $\qquad x_{op}[n] \equiv [\underset{\uparrow\, n=0}{j} \;\; -1 \;\; 3j \;\; \underset{\uparrow\, n=N-1}{1}] \equiv b[n]$

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b) Given that $Y[K] = A[K]\,B[K]$ then, according to the DFT properties, $y[n] = a[n] \circledast b[n]$, where $\circledast$ denotes circular convolution.

Thus: $$y[n] = a[n] \circledast b[n] = b[n] \circledast a[n] = \sum_{l=0}^{N-1} b[l]\, a[(n-l)_N]$$

$$\begin{array}{l|cccc|l}
b[l]: & j & -1 & 3j & 1 & j \ldots \\
 & \uparrow & & & \uparrow & \\
 & l=0 & & & l=N-1 & \\
n=0,\ a[(-l)_N]: & 0 & 3 & 0 & 3 & 0 \ldots \quad \therefore \sum_l = 0 \\
n=1,\ a[(1-l)_N]: & 3 & 0 & 3 & 0 & 3 \ldots \quad \therefore \sum_l = 3j + 9j = 12j \\
n=2,\ a[(2-l)_N] & 0 & 3 & 0 & 3 & 0 \ldots \quad \therefore \sum_l = 0 \\
n=3,\ a[(3-l)_N] & 3 & 0 & 3 & 0 & 3 \ldots \quad \therefore \sum_l = 3j + 9j = 12j
\end{array}$$

hence: $y[n] = \text{ifft}\{Y[K]\} \equiv [0 \quad 12j \quad 0 \quad 12j]$

c) Given that $A[K] = \mathcal{Re}\{X[K]\} = \dfrac{X[K] + X^*[K]}{2}$

and $B[K] = j\,\mathcal{Im}\{X[K]\} = j\,\dfrac{X[K] - X^*[K]}{2j}$

then, $$Y[K] = A[K]\,B[K] = \frac{(X[K] + X^*[K])(X[K] - X^*[K])}{4} = \frac{X[K]X[K] - X^*[K]X^*[K]}{4}$$

$= Z[K]$, which means that $y[n] = z[n]$.

Another way of showing this result is:

$$x[n] \xleftrightarrow{DFT} X[K]$$
$$x^*[(-n)_N] \longleftrightarrow X^*[K]$$

and,

$$Z[K] = \frac{X[K]X[K] - X^*[K]X^*[K]}{4} \xleftrightarrow{DFT} \frac{x[n] \circledast x[n] - x^*[(-n)_N] \circledast x^*[(-n)_N]}{4} =$$

$$= \frac{x[n] + x^*[(-n)_N]}{2} \circledast \frac{x[n] - x^*[(-n)_N]}{2}$$

$$= x_{ep}[n] \circledast x_{op}[n] = a[n] \circledast b[n] = y[n]$$

which means that $y[n] = z[n]$.

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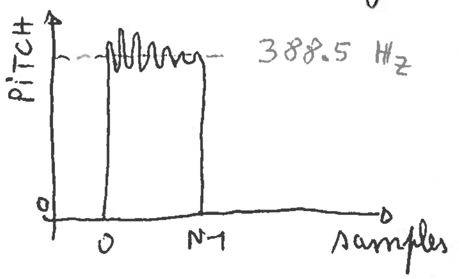
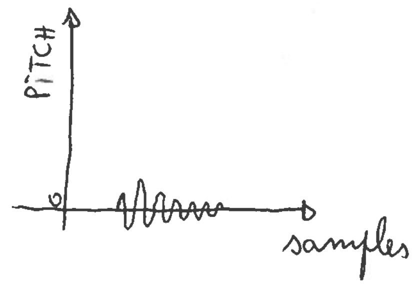
4. In this exercise, it is important to realize that the two signals processed by the two pieces of code look like:





a) This means that in the first case the signal looks like a "boxcar" function having small variations around its average value (388.5 Hz) that represents (the average value, or D.C. component) the dominant component of the overall signal and, therefore, its autocorrelation has a triangular shape that does not reveal the periodicity of the small variations.
Only when the D.C. component is subtracted from the signal is that the small pitch variations (i.e., the vibrato) have the opportunity to stand out and, as a consequence, the autocorrelation function is able to reflect their periodicity

This means that the first piece of code generates Figure B, and the second piece of code generates Figure C.

b) The autocorrelation function in Figure C reveals that a relevant periodicity exist in the pitch for lag = 8 samples, which means that the average period of the vibrato (i.e., the variation of the pitch signal) is $8 \times \frac{512}{22050} \simeq 0.18576$ s and its reciprocal, the vibrato frequency, is $\frac{1}{0.18576} \simeq 5.4$ Hz.

c) Taking into consideration the DFT properties:

$$x[l] \longleftrightarrow X[k]$$

$$x^*[(-l)_N] \longleftrightarrow X^*[k]$$

$$r_{xp}[l] = x[l] \circledast x^*[(-l)_N] \longleftrightarrow X[k]X^*[k] = |X[k]|^2 = R_{xp}[k]$$

where $\circledast$ denotes circular convolution.

However, we are interested in the autocorrelation definition that presumes the linear convolution:

$$r_x[l] = x[l] * x^*[-l] \longleftrightarrow X(e^{j\omega})\, X^*(e^{j\omega}) = |X(e^{j\omega})|^2 = R_x(e^{j\omega})$$

In order for $r_x[l] = r_{xp}[l]$ within one period ($l = 0, 1, \ldots, N-1$) after taking $R_{xp}[k] = R_x(e^{j\omega})\Big|_{\omega = k\frac{2\pi}{N},\ k=0,1,\ldots,N-1}$, we must make sure that the circular convolution delivers the same result of the linear convolution. If the length of $x[l]$ is L, then that is achieved by choosing $N \geq 2L-1$ after zero-padding. On this assumption, a block diagram of the frequency-domain processing delivering the desired result is:

$$x[l] \xrightarrow{L} \boxed{\text{zero-padding}} \xrightarrow{N} \boxed{\begin{matrix}\text{DFT}\\(N)\end{matrix}} \xrightarrow{N} \boxed{|\cdot|^2} \xrightarrow{N} \boxed{\begin{matrix}\text{IDFT}\\(N)\end{matrix}} \xrightarrow{N} r_x[l]$$

where $L = 46$ and $N \geq 91$, for example, $N = 128$ (a power of 2 number)

$\boxed{5.}$ **a)** When the DFT/FFT is looked at as a filter bank, the frequency response of each filter depends on the window that multiplies the data. In the case of the Rectangular window, the pass-band is narrow (improving spectral selectivity) but the main-to-side lobe attenuation is poor causing high levels of out-of-band leakage, also known as far-end leakage. In the case of the Hanning window, the pass-band is wide (causing the representation of sinusoids in a spectrogram to be thicker) aggravating near-end leakage but because the main-to-side lobe attenuation is improved, the far-end leakage is smaller and, as a consequence, spectrograms become "cleaner". Thus, spectrogram A has been generated using the Hanning window and spectrogram B has been generated using the Rectangular window.

b) The audio signal contains:

→ a frequency-modulated sinusoid whose center frequency is 6 KHz, the frequency deviation around the mean is about 1 KHz and the frequency of the frequency-modulation is about $\frac{1}{1.1-0.4} \simeq 1.43\,Hz$; this signal exists for $t \geq 0.05$ s and $t \leq 1.45$ s.

→ a sinusoid whose frequency increases linearly between 0 Hz and 4 KHz for $t \geq 0$ s and $t \leq 0.4$ s.

→ another frequency-modulated sinusoid that exists for $t \geq 0.5$ and $t \leq 1$ s. its center frequency is 2 KHz, the frequency deviation is around 1 KHz and the frequency of the frequency modulation is around $\frac{1}{0.9-0.6} \cong 3{,}3\,Hz$.

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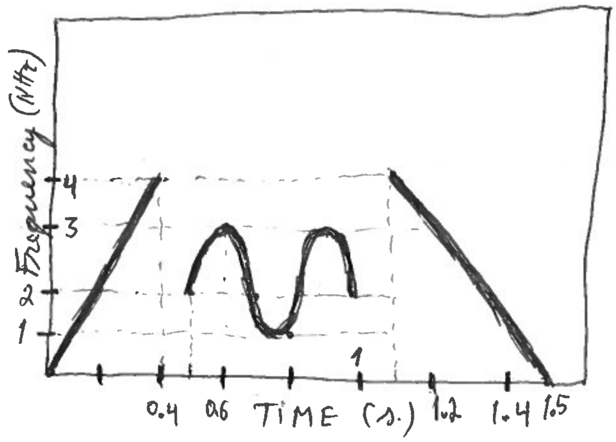
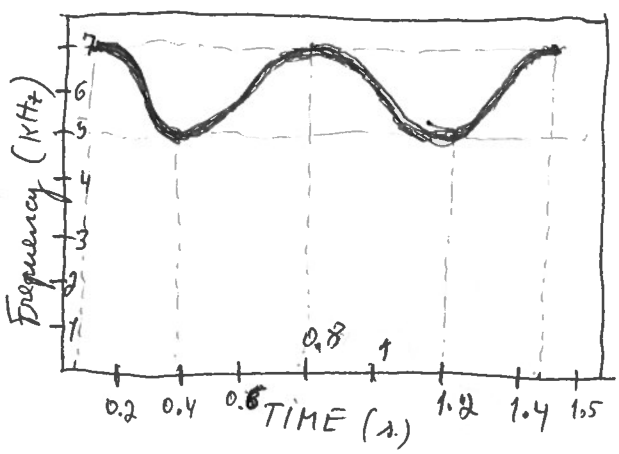
→ another sinusoid whose frequency decreases linearly between 4 KHz and 0 Hz for $t \geq 1.1$ s and $t \leq 1.5$ s.

e) If a line is drawn in the spectrogram separating frequencies below 4.5 KHz (as a consequence of low-pass filtering) and frequencies above 4.5 KHz (as a consequence of ideal high-pass filtering), then the following two spectrograms are easily obtained:

→ after ideal low-pass filtering:



→ and after ideal high-pass filtering:



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